Some lines and planes of tetrahedron

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Let Th = ABCD be an arbitrary tetrahedron, see Fig. 1.

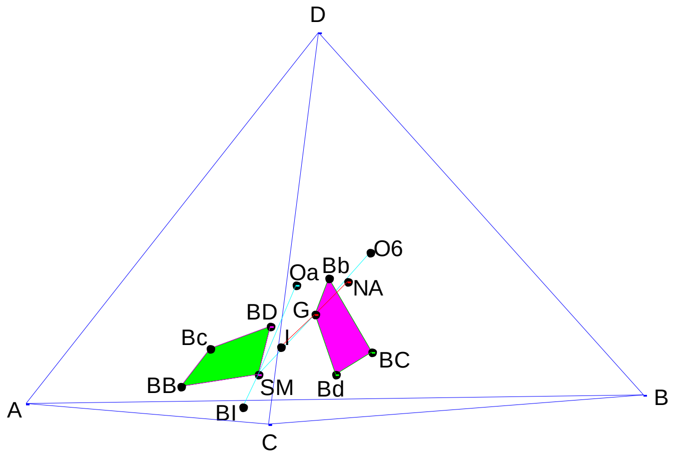

Fig. 1

We define the following points:

$I=(S_a:S_b:S_c:S_d)$, the incenter of Th in barycentric coordinates, where

S_a, S_b, S_c, S_d are areas of the respective faces of Th.

$G=(1:1:1:1)$, the centroid of Th.

$SM=(S_a{}^2:S_b{}^2:S_c{}^2:S_d{}^2)$, the algebraic analogue of symmedian point X(6).

$NA=(S_b+S_c+S_d-S_a:*:*:*)$, the algebraic analogue of the Nagel Point X(8), where $*$ means cyclicly.

$O6=(S_b{}^2+S_c{}^2+S_d{}^2-S_a{}^2:*:*:*)$, the algebraic analogue of X(69).

$Oa=(S_a{}^2\cdot(S_b{}^2+S_c{}^2+S_d{}^2-S_a{}^2):*:*:*)$, the algebraic analogue of circum-center X(3).

$$BI = (S_a^4 : S_b^4 : S_c^4 : S_d^4) \ .$$

$$Bb = (\frac{1}{S_b^2} : \frac{1}{S_c^2} : \frac{1}{S_d^2} : \frac{1}{S_a^2})$$, Brocard points like analogue.

$$Bc = (\frac{1}{S_c^2} : \frac{1}{S_d^2} : \frac{1}{S_a^2} : \frac{1}{S_b^2})$$, Brocard points like analogue.

$$Bd = (\frac{1}{S_d^2} : \frac{1}{S_a^2} : \frac{1}{S_b^2} : \frac{1}{S_c^2})$$, Brocard points like analogue.

$$BB = (S_a^2 \cdot S_b^2 : S_b^2 \cdot S_c^2 : S_c^2 \cdot S_d^2 : S_d^2 \cdot S_a^2)$$, isogonal conjugate of Bb.

$$BC = (S_a^2 \cdot S_c^2 : S_b^2 \cdot S_d^2 : S_c^2 \cdot S_a^2 : S_d^2 \cdot S_b^2)$$, isogonal conjugate of Bc.

$$BD = (S_a^2 \cdot S_d^2 : S_b^2 \cdot S_a^2 : S_c^2 \cdot S_b^2 : S_d^2 \cdot S_c^2)$$, isogonal conjugate of Bd.

The following triples of points {I,G,NA}, {SM,G,O6}, {Oa,SM,BI} are collinear respectively. Let $M_1 = [I; G; NA]$ be the 3x4 matrix formed by the coordinates of points I,G,NA, then we have

$$\det([M_1(:,1), M_1(:,2), M_1(:,3)]) = 0 \ ,$$

$$\det([M_1(:,1), M_1(:,2), M_1(:,4)]) = 0 \ ,$$

$$\det([M_1(:,1), M_1(:,3), M_1(:,4)]) = 0 \ ,$$

$$\det([M_1(:,2), M_1(:,3), M_1(:,4)]) = 0 \ ,$$

where for example $M_1(:,1)$ is the first column of M_1.

All these three lines have their analogue for an arbitrary triangle.

Furthermore, the following quadruples of points {G,Bd,BC,Bb} and {SM,BD,Bc,BB} are complanar respectively[1]. This means

$$\det([G; Bd; BC; Bb]) = 0$$, and

$$\det([SM; BD; Bc; BB]) = 0 \ .$$

[1] Avni Pllana: http://trisectlimacon.webs.com/MiscTetra4.pdf