

Some results on side-bisector reflected triangle

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Let ABC be an arbitrary triangle. Let A', B', C' be the reflections of vertices A, B, C on the bisectors of sides BC, CA, AB respectively. Points A', B', C' form a new triangle $A'B'C'$ and have barycentric coordinates as follows

$$\begin{aligned} A' &= (a^2 : c^2 - b^2 : b^2 - c^2) \\ B' &= (c^2 - a^2 : b^2 : a^2 - c^2) . \\ C' &= (b^2 - a^2 : a^2 - b^2 : c^2) \end{aligned} \quad (1)$$

The lines through vertices A', B', C' and perpendicular respectively to the sides BC, CA, AB of triangle ABC , are concurrent at the *de Longchamps point* $X(20)$ of triangle ABC , see Fig.1.

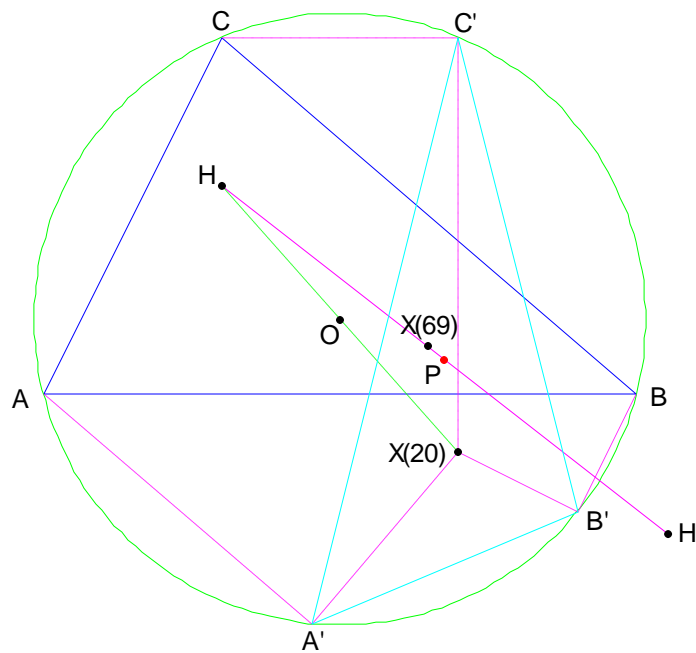


Fig. 1

Let A'_p, B'_p, C'_p be the orthogonal projections of vertices A', B', C' respectively to the sides BC, CA, AB of triangle ABC . Then lines AA'_p, BB'_p, CC'_p are concurrent at the point $X(69)$ of triangle ABC .

¹ Math Forum Public Discussions:

<http://mathforum.org/kb/thread.jspa?forumID=129&threadID=2249511&messageID=7422634#7422634>

The *Simson lines* of vertices A', B', C' with respect to triangle ABC , are concurrent at the point $P = (u : v : w)$, where

$$u = a^2 \cdot S_a \cdot [2 \cdot a^2 \cdot b^2 \cdot c^2 + b^2 \cdot S_b \cdot (b^2 - a^2) + c^2 \cdot S_c \cdot (c^2 - a^2)], \quad (2)$$

where $S_a = b^2 + c^2 - a^2$, $S_b = c^2 + a^2 - b^2$, $S_c = a^2 + b^2 - c^2$.

Let H, H' be the orthocenters of triangles $ABC, A'B'C'$ respectively. Then points $H, X(69), P, H'$ are collinear, and $\overline{HP} = \overline{PH'}$, see Fig 1.

Now we can repeat the construction of point P^2 for the triangle $A'B'C'$ as starting triangle, and so on, after 10000 iterations we obtain a distribution of points P as shown in Fig. 2, the red points.

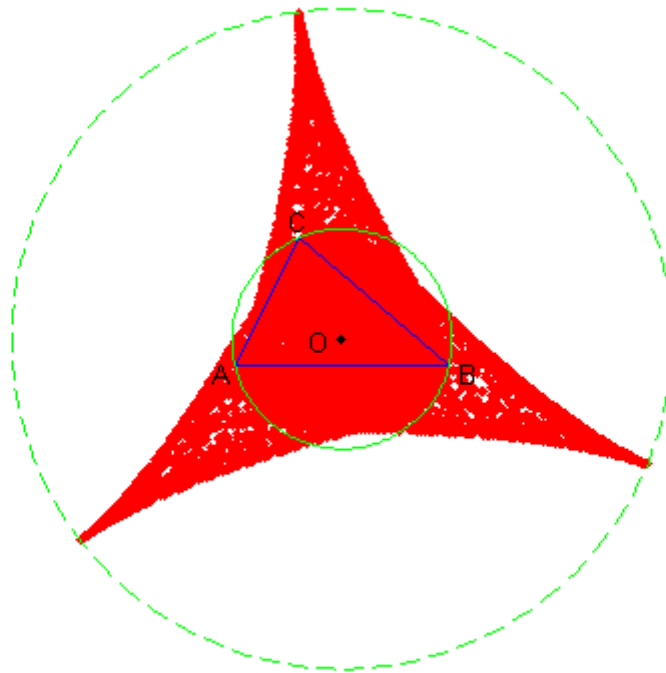


Fig. 2

The red P points lie inside the circle with center at the orthocenter O of triangle ABC , and radius $3 \cdot R$, where R is the circumradius of triangle ABC .

Next we find the envelope of the red P points. We start from the equations of the *deltoid*³

² This point is listed in Encyclopedia of Triangle Centers as point $X(5562)$ with the false information that it has been first discovered in October 2, 2013.

³ Deltoid: <http://mathworld.wolfram.com/Deltoid.html>

$$\begin{aligned} x &= 3R\left[\frac{2}{3}\cos(t) + \frac{1}{3}\cos(2t)\right] \\ y &= 3R\left[\frac{2}{3}\sin(t) - \frac{1}{3}\sin(2t)\right] \end{aligned} \quad (3)$$

The slope of deltoid (3) for an arbitrary angle $t = \varphi$, is

$$\frac{dy}{dx} = \frac{dy/d\varphi}{dx/d\varphi} = -\tan\frac{\varphi}{2} \quad (4)$$

Let $D_1 = [x(\varphi), y(\varphi)]$ and $D_2 = [x(\varphi + \pi), y(\varphi + \pi)]$ be two points in deltoid (3). The line D_1D_2 has the slope $s = \tan(\varphi)$, or $s = -\tan(-2\varphi/2)$. This means that the line D_1D_2 has the same slope as the tangent of deltoid (3) at the point $D_3 = [x(-2\varphi), y(-2\varphi)]$. The points D_1, D_2, D_3 are collinear, because

$$\begin{vmatrix} x(\varphi) & y(\varphi) & 1 \\ x(\varphi + \pi) & y(\varphi + \pi) & 1 \\ x(-2\varphi) & y(-2\varphi) & 1 \end{vmatrix} = 0 \quad (5)$$

Now we define the point $D_4 = (D_1 + D_3)/2$, so we have

$$\begin{aligned} x &= \frac{3R}{2}\left[\frac{2}{3}\cos(\varphi) + \cos(2\varphi) + \frac{1}{3}\cos(4\varphi)\right] \\ y &= \frac{3R}{2}\left[\frac{2}{3}\sin(\varphi) - \sin(2\varphi) + \frac{1}{3}\sin(4\varphi)\right] \end{aligned} \quad (6)$$

The curve defined by equations (6) we call the *bell deltoid*, see Fig. 3.

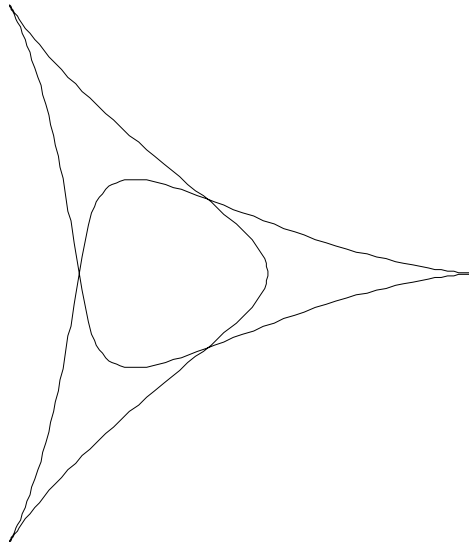


Fig. 3

Finally the envelope of the red points P is a *bell deltoid* with orientation opposite to the *first Morley triangle*⁴, and it is shown in Fig. 4.

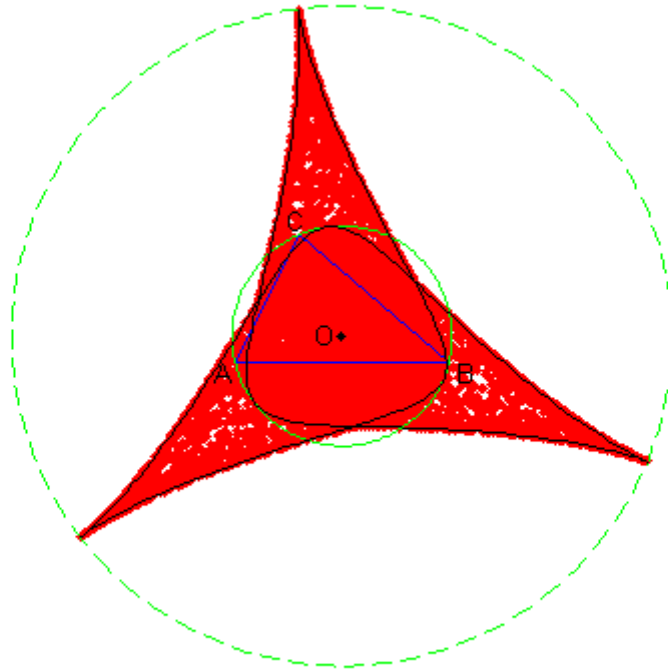


Fig. 4

⁴ Simson Line: <http://mathworld.wolfram.com/SimsonLine.html>