Some results on orthopolars for a given point

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First published: 28.09.2015

Constructing the orthopole[1] P for a given line L is quite easy, however the inverse problem is considerably difficult. A solution to this problem was given by Jean-Pierre Ehrmann[2] (2006), and in this article we use a different approach. In Fig. 1 is shown the orthopole construction for an arbitrary triangle ABC and an arbitrary line L.

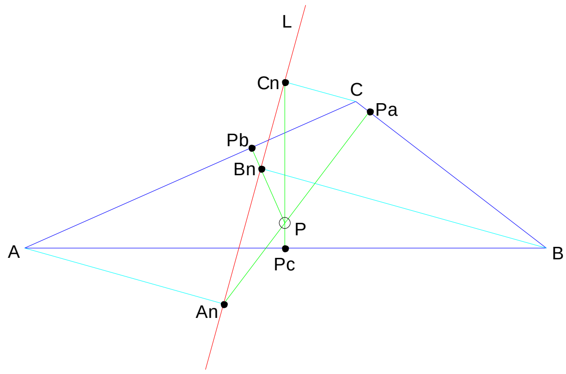

Fig. 1

The problem is how to construct the orthopolar L for a given point P. In general there is no ruler and compass construction, but we can handle the problem algebraically.

The lines AAn, BBn, CCn in Fig. 1 are parallel and intersect the line at infinity at the same point Q = (x : y : z) , that satisfies the equation

$$x + y + z = 0 , \tag{1}$$

or in normalized form

$$X + Y + 1 = 0 , \tag{2}$$

[1] Jean-Pierre Ehrmann: http://forumgeom.fau.edu/FG2004volume4/FG200405.pdf

[2] Jean-Pierre Ehrmann: http://forumgeom.fau.edu/FG2006volume6/FG200638.pdf

where X = x/z and Y = y/z .

Points Pa, Pb, Pc are orthogonal projections of point P on the respective sides of triangle ABC. Lines AQ, BQ, CQ intersect the lines PPa, PPb, PPc respectively at points An, Bn, Cn, which have to be collinear. From the collinearity condition of points An, Bn, Cn and from equation (2) we obtain a cubic equation in X. It has three real solutions when point P is inside the deltoid[3] corresponding to triangle ABC. When point P is outside the deltoid, there is only one real solution.

The isogonal conjugate of a solution point $Q_j = (X_j : Y_j : 1)$ is a point T_j on the circumcircle of triangle ABC. The Simson line of point T_j is orthogonal to the lines AQ_j, BQ_j, CQ_j, and this means it is parallel to the sought orthopolars L_j of the given poin P.

The lines through point P and tangent to the deltoid have exactly the sought directions of the lines AQ_j, BQ_j, CQ_j, see Fig. 2.

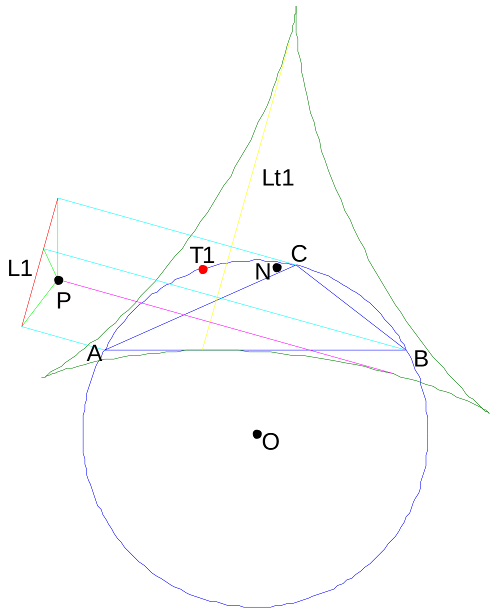

Fig. 2

For the sake of clarity in Fig. 2 is shown the case when the given point P is outside the deltoid. Point N is the center of deltoid, it is the midpoint of segment OH, where H is the orthocenter of triangle ABC.

[3] Avni Pllana: http://trisectlimacon.webs.com/Deltoid_Orthopole.pdf