Three points on incircle

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Problem:

Let ABC be an arbitrary triangle. Construct points D, E, F on the incircle of triangle ABC, such that triples {C, D, E}, {B, E, F}, {A, F, D} are collinear respectively.

Solution:

Let Ga, Gb, Gc be the points where the respective sides of triangle ABC tangent the incircle, see Fig. 1. Lines GcGb and BC intersect at point Pa. Lines GcGa and AC intersect at point Pb. The line through points Pa, Pb is the tripolar line of Gergone point Ge, or X(7). Let Ak be the intersection point of lines GcGb and AGa. Let Bk be the intersection point of lines GcGa and BGb. Then line PaBk intersects the incircle at points D1 and E, and line PbAk intersects the incircle at points D and F1.

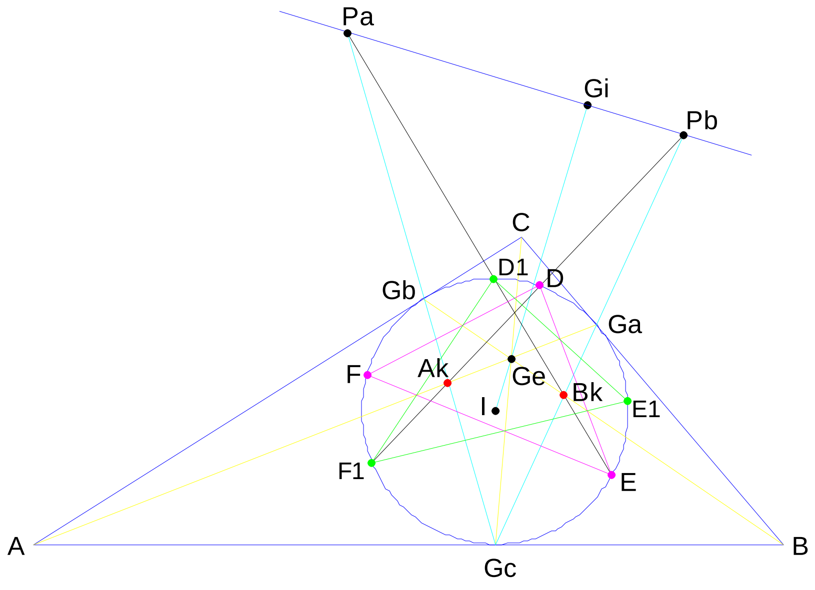

Fig. 1

The triples of points {C, D, E} and {C, D1, F1} are collinear respectively. Lines AD and BE intersect at point F that also lies on incircle. Lines AF1 and BD1 intersect at point E1 that also lies on incircle. This is the complete solution of the problem.

We observe that the tripolar line of Ge is the same as polar line that corresponds to Ge with respect to the incircle. Point Gi in Fig.1 is the inverse of Ge.

Let $Ge = (g_1 : g_2 : g_3)$ in barycentric coordinates, where $g_1 = 1/(b + c - a)$, $g_2 = 1/(c + a - b)$, $g_3 = 1/(a + b - c)$. Let u, v, w be the barycentric coordinates of point D1, then they are the solution of the following system of equations

$$u + v + w = 1 \quad , \qquad (1)$$
$$v \cdot w \cdot g_1^2 = u^2 \cdot g_2 \cdot g_3 \ , \qquad (2)$$
$$3 \cdot g_2 \cdot g_3 \cdot u = v \cdot g_3 \cdot g_1 + w \cdot g_1 \cdot g_2 \ , \qquad (3)$$

where (1) is the equation of absolute coordinates, equation (2) is derived from the fact that lines AA1, BD1 and CGc are concurrent, where A1 is the intersection of lines CD1 and BGb. Equation (3) is derived from the fact that point D1 lies on the line PaBk.

Knowing u, v, w, for the point D we have

$$D = (v^2 \cdot g_1^3 \ : \ u \cdot v \cdot g_1 \cdot g_2^2 \ : \ u^2 \cdot g_2^2 \cdot g_3) \ . \qquad (4)$$

Equation (4) is derived from the fact that lines CD and CD1 are iso-Ge conjugate[1] to each other.

Since the entire construction is projective, it is the same for the other conic sections instead of incircle as a circle. This means we can take any other conic section, choose any three tangents that form a triangle ABC, and seek points D, E, F on that conic section with above properties.

[1] Avni Pllana: http://trisectlimacon.webs.com/MiscTetra4.pdf