An interesting connection between Gegenbauer and Legendre polynomials

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It is well-known that Legendre polynomials[1] are a special case of Gegenbauer polynomials[2], namely for $\alpha = \frac{1}{2}$, where Gegenbauer polynomials are given as follows,

$$
\begin{aligned}
&C_0^{(\alpha)}(x) = 1 ,\\
&C_1^{(\alpha)}(x) = 2\alpha x ,\\
&C_2^{(\alpha)}(x) = -\alpha + 2\alpha(1+\alpha)x^2 ,\\
&C_3^{(\alpha)}(x) = -2\alpha(1+\alpha)x + \frac{4}{3}\alpha(1+\alpha)(2+\alpha)x^3 ,\\
&\qquad\vdots\\
&nC_n^{(\alpha)}(x) = 2(n+\alpha-1)xC_{n-1}^{(\alpha)}(x) - (n+2\alpha-2)C_{n-2}^{(\alpha)}(x) .
\end{aligned} \tag{1}
$$

In this article will be shown another connection between Gegenbauer and Legendre polynomials using the following approach.

Let us consider the line segment $[-1,1]$, with two punctual unit charges fixed at points $x=-1$, and $x=1$. For the sake of simplicity let us put on the segment $[-1,1]$ three, free moving punctual unit charges, that settle at some equilibrium points, due to a repelling force between them, defined as

$$F_{12} = \frac{1}{x_2 - x_1} , \tag{2}$$

where x_1, x_2 are respective positions of two unit charges, see Fig. 1.

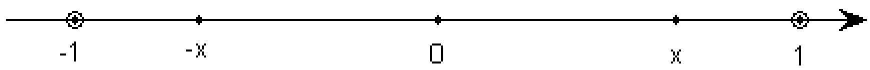

Fig. 1

[1] Legendre polynomials: https://en.wikipedia.org/wiki/Legendre_polynomials
[2] Gegenbauer polynomials: https://en.wikipedia.org/wiki/Gegenbauer_polynomials

According to equation (2) and Fig. 1, we can write

$$\frac{1}{1-x} = \frac{1}{x} + \frac{1}{2x} + \frac{1}{1+x} \ . \tag{3}$$

After some arrangements from (3) we obtain

$$7x^2 - 3x = 0 \ . \tag{4}$$

The polynomial on the left side of (4) is up to a scaling factor the third degree Gegenbauer polynomial for $\alpha = \frac{3}{2}$. This means that the roots of (4) are the equilibrium points of the three unit charges.

This way we obtain the first few Gegenbauer polynomials for $\alpha = \frac{3}{2}$,

$$\begin{aligned}
C_0^{\left(\frac{3}{2}\right)}(x) &= 1 \,, \\
C_1^{\left(\frac{3}{2}\right)}(x) &= x \,, \\
C_2^{\left(\frac{3}{2}\right)}(x) &= \frac{1}{4}(5x^2 - 1) \,, \\
C_3^{\left(\frac{3}{2}\right)}(x) &= \frac{1}{4}(7x^3 - 3x) \,, \\
C_4^{\left(\frac{3}{2}\right)}(x) &= \frac{1}{8}(21x^4 - 14x^2 + 1) \,, \\
C_5^{\left(\frac{3}{2}\right)}(x) &= \frac{1}{8}(33x^5 - 30x^3 + 5x) \,,
\end{aligned} \tag{5}$$

standardized under condition $C_n^{\left(\frac{3}{2}\right)}(1) = 1$. The corresponding recurrence formula is

$$(n+3)C_{n+1}^{\left(\frac{3}{2}\right)}(x) = (2n+3)xC_n^{\left(\frac{3}{2}\right)}(x) - nC_{n-1}^{\left(\frac{3}{2}\right)}(x) \ , \quad n = 1, 2, \cdots. \tag{6}$$

Now let us determine the equilibrium points between the three stable unit charges and the two fixed unit charges at the ends of the line segment $[-1, 1]$, see Fig. 2.

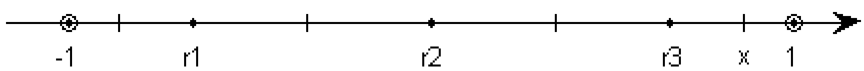

Fig. 2

According to equation (2) and Fig. 2, we have

$$\frac{1}{1-x}=\frac{1}{x-r_3}+\frac{1}{x-r_2}+\frac{1}{x-r_1}+\frac{1}{1+x} \ , \tag{7}$$

where r_1, r_2, r_3 are the roots of (4).

After some arrangements from (7) we obtain

$$35x^4-30x^2+3=0 \ . \tag{8}$$

The left side of (8) is the fourth degree Legendre polynomial, and the roots of (8) are the sought equilibrium points. This way we obtain the corresponding Legendre polynomial for a different number of unit charges.