Three points on arbitrary circle[1]

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Problem 1:

Let ABC be an arbitrary triangle. Construct points Ar, Br, Cr on arbitrary circle with center P and radius r, such that triples {A, Br, Cr}, {B, Cr, Ar}, {C, Ar, Br} are collinear respectively.

Solution:

Let Ai, Bi, Ci be the poles[2] of respective sides BC, CA, AB of triangle ABC with respect to arbitrary circle Circ(P,r), see Fig. 1. Triangles ABC and AiBiCi are perspective, and their perspectivity point is Gp. Lines AAi, BBi, CCi and BiCi, CiAi, AiBi intersect respectively at points Ap, Bp, Cp. Lines BpCp, CpAp, ApBp intersect Circ(P,r) at points {Ar, Al}, {Br, Bl}, {Cr, Cl} respectively, forming the sought right triangle ArBrCr and left triangle AlBlCl.

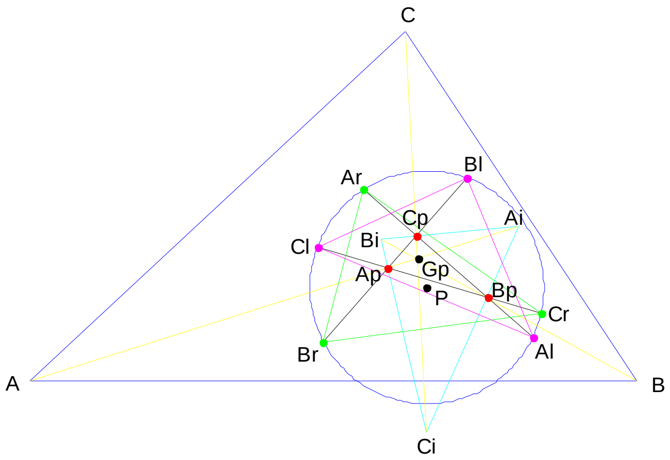

Fig. 1

Let Am, Bm, Cm be the middle points of respective sides of triangle ABC. The lines through Am, Bm, Cm and orthogonal to the respective sides of triangle AiBiCi are concurrent. This also holds for tetrahedron, with planes through side middle points and orthogonal to the respective sides of inverted tetrahedron with respect to some arbitrary sphere.

[1] Avni Pllana: Three points on incircle. http://trisectlimacon.webs.com/incircle3p.pdf
[2] Pole and polar: http://en.wikipedia.org/wiki/Pole_and_polar

Problem 2: Three points on skew triangle

Let ABC be an arbitrary triangle. Let A1B1C1 be an arbitrary triangle inside triangle ABC, such that triples of points {A,C1,B1}, {B,A1,C1}, {C,B1,A1} are collinear respectively. Construct points A2, B2, C2, on the respective sides of triangle A1B1C1, such that {A,B2,C2}, {B,C2,A2}, {C,A2,B2} are respectively collinear.

Solution:

Let lines CC1 and CB1 intersect line AB at points Ca and Cb respectively, see Fig. 2. Let the line through the sought points A2, B2 intersect line AB at point Ck. We observe that Ca = (p:q) and Cb = (u:v) are isoconjugate with respect to Ck = (x:y). This means $x = \sqrt{p \cdot u}$, and $y = \sqrt{q \cdot v}$. In order to construct point Ck we draw the line through Ca and parallel to line BC, that intersects line CA at point Bc. We draw the line through Cb and parallel to line CA, that intersects line BC at point Ac. Now lines AAc and BBc intersect at point Pc. The line CPc intersects line AB at point $Cp = (p \cdot u : q \cdot v)$. The line through Cp and orthogonal to line AB intersects the semicircle with diameter AB at point Qc. Now the bisector of angle(A,Qc,B) intersects line AB at the sought point Ck.

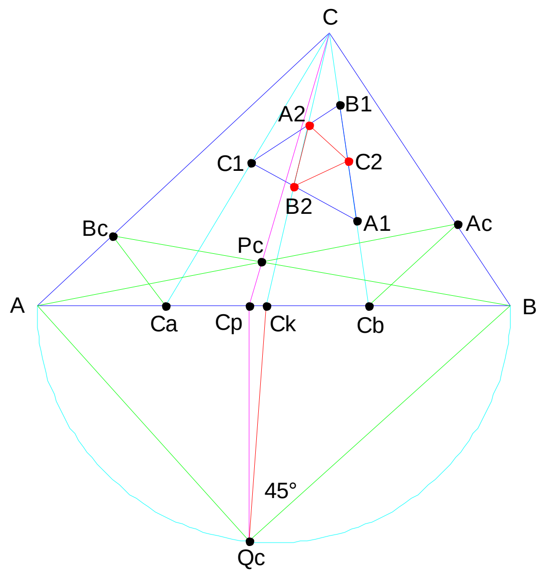

Fig. 2

Circumcircles of triangles A1BC, B1CA, C1AB are concurrent.

Problem 3: Three points on cevian triangle[3]

Let ABC be an arbitrary triangle. Construct points Ar, Br, Cr on the cevian triangle of arbitrary point P, such that triples {A, Br, Cr}, {B, Cr, Ar}, {C, Ar, Br} are collinear respectively.

Solution:

Let A1B1C1 be the cevian triangle of point P, see Fig. 3. The line through C1 and parallel to line AC intersects line BC at point Ca. The line through C1 and parallel to line BC intersects line AC at point Cb. On the segment CaCb we construct points Ga and Gb such that

$$\frac{\overline{CaGb}}{\overline{CaCb}} = \frac{\overline{CbGa}}{\overline{CbCa}} = \frac{\sqrt{5}-1}{2},$$

that is the golden ratio[4].

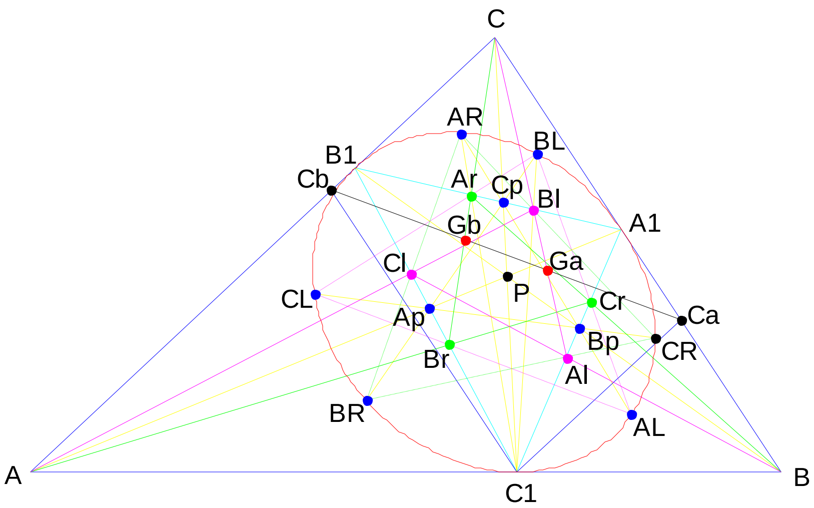

Fig. 3

The line CGb intersects lines A1B1, B1C1 at the required points Ar, Br respectively.

[3] Cevian triangle: http://mathworld.wolfram.com/CevianTriangle.html

[4] Golden ratio: http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/phi2DGeomTrig.html

We also can construct points AR, BR, CR with above properties on the inellipse corresponding to the cevian triangle A1B1C1, the red ellipse in Fig. 3. Let Ap, Bp, Cp be the intersection points of the lines B1C1, C1A1, A1B1 with respective cevians of point P. Then lines BpCp and C1Ar intersect at the required point AR on the inellipse.

In Fig. 4 is shown how a general isoconjugation can be reduced to an isotomic conjugation[5]. Let Cp = (u : v) be an arbitrary point on the line AB. The line through Cp and parallel to line AC intersects line BC at point Ca. The line through Cp and parallel to line BC intersects line AC at point Cb. The line CCp intersects the segment CaCb at the middle point M.

Now let Cq = (x : y) be another arbitrary point on the line AB. Line CCq intersects line CaCb at point Tq. We construct point Tk on the line CaCb such that $\overline{\mathrm{MTq}} = \overline{\mathrm{MTk}}$. Now line CTk intersects line AB at point $\mathrm{Ck} = (u^2/x : v^2/y)$. This means that points Cq and Ck are isoconjugate with respect to point Cp.

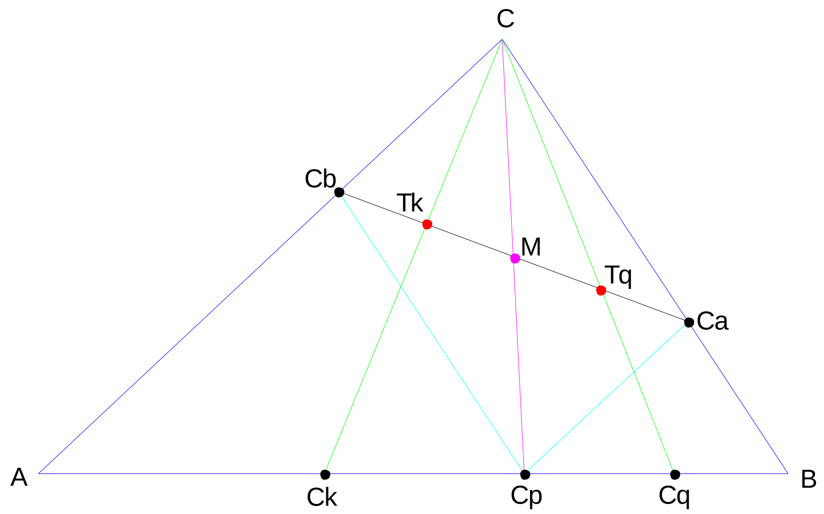

Fig. 4

[5] Isotomic conjugate: http://en.wikipedia.org/wiki/Isotomic_conjugate