Angle Trisection using Limacon of Pascal

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A dozen of angle trisection methods using limacon of Pascal[1] are known, and one more such a method is shown in Fig.1.

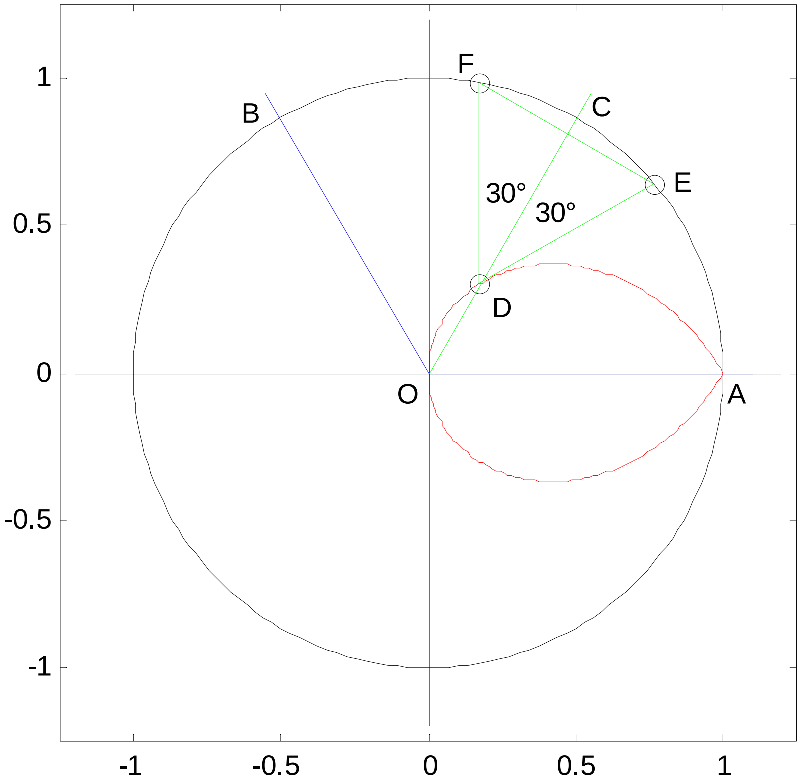

Fig.1

Let $\angle AOB = \alpha$ be the given angle to be trisected. We draw line OC which bisects $\angle AOB$, therefore $\angle AOC = \beta = \frac{\alpha}{2}$. Line OC intersects the limacon (red loop) at point D.

Limacon of Pascal[2] is defined in the polar coordinates by the equation

$$r = \frac{1}{2} + \cos\theta \quad . \tag{1}$$

We use only the little loop of the limacon (1) , enlarged by factor 2, mirrored with respect to the y-axis, and shifted 1 unit to the right along the x-axis. Such a transformed limacon is defined in the rectangular coordinates by the following parametric equations

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[1] Loy, J. "Trisection of an Angle." `http://www.jimloy.com/geometry/trisect.htm`

[2] Eric W. Weisstein. "Limaçon." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/Limacon.html

$$\begin{aligned} x &= 1-(1+2\cos\theta)\cos\theta \\ y &= (1+2\cos\theta)\sin\theta \end{aligned} \quad , \tag{2}$$

where $\theta = \frac{4\pi}{3} - \frac{2}{3}\beta$, and $\beta = \frac{\alpha}{2}$ is the bisector of the given angle $\alpha \in [\,0, 2\pi]$ to be trisected.

At point D on both sides of line OC we draw lines DE and DF which make $30°$ with line OC. Points E and F trisect $\angle AOB$, and ΔDEF is equilateral.

In order to prove the above statements it suffices to show that $\angle EOC = \frac{\alpha}{6}$. Let us denote $\angle EOC$ by γ, and $\overline{OD}$ by r. The Law of Sines for ΔOED yields

$$\frac{1}{\sin\left(\pi - \frac{\pi}{6}\right)} = \frac{r}{\sin\left(\frac{\pi}{6} - \gamma\right)} \quad . \tag{3}$$

From equations (2) and relation $\theta = \frac{4\pi}{3} - \frac{\alpha}{3}$, follows

$$r = \sqrt{x^2 + y^2} = \sqrt{2 + 2\cos\left(\frac{4\pi}{3} - \frac{\alpha}{3}\right)} \quad . \tag{4}$$

From (3) and (4) follows

$$2\sin^2\left(\frac{\pi}{6} - \gamma\right) = 1 + \cos\left(\frac{4\pi}{3} - \frac{\alpha}{3}\right) \ ,$$

$$2\sin^2\left(\frac{\pi}{6} - \gamma\right) = \cos^2\left(\frac{\pi}{6} - \gamma\right) + \sin^2\left(\frac{\pi}{6} - \gamma\right) + \cos\left(\frac{4\pi}{3} - \frac{\alpha}{3}\right) \ ,$$

$$\sin^2\left(\frac{\pi}{6} - \gamma\right) - \cos^2\left(\frac{\pi}{6} - \gamma\right) = \cos\left(\frac{4\pi}{3} - \frac{\alpha}{3}\right) \ ,$$

$$-\cos\left(\frac{\pi}{3} - 2\gamma\right) = \cos\left(\frac{4\pi}{3} - \frac{\alpha}{3}\right) \ ,$$

$$\cos\left(\pi + \frac{\pi}{3} - 2\gamma\right) = \cos\left(\frac{4\pi}{3} - \frac{\alpha}{3}\right) \ ,$$

and finally we obtain $\gamma = \frac{\alpha}{6}$.

The presented trisection method can be generalized for the angle n-section, see Fig.2.

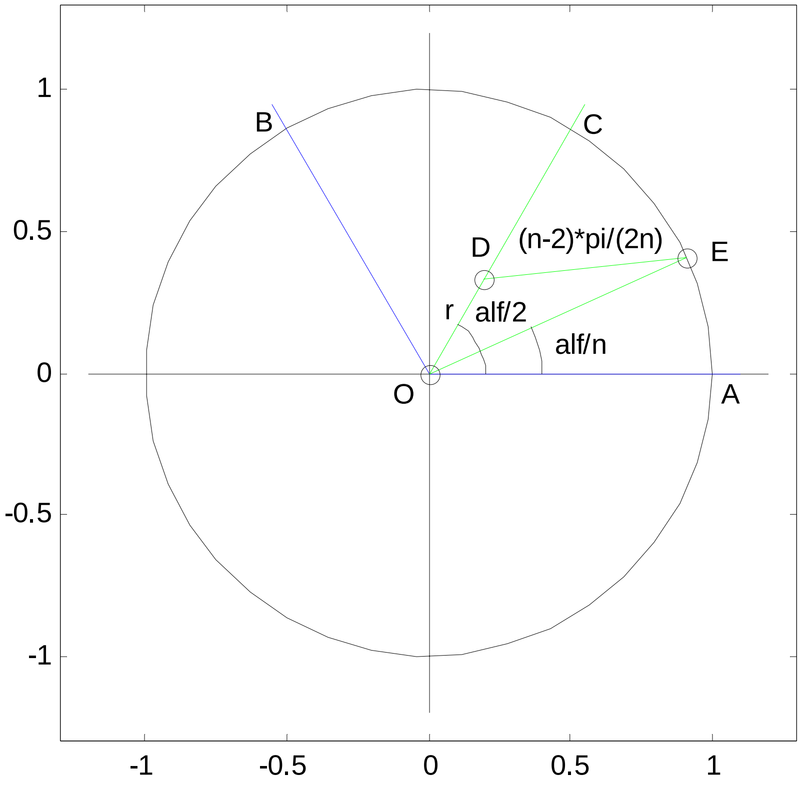

Fig.2

In Fig.2 angle $\angle EDC$ is the half of the vertex angle of a regular n-gon. The Law of Sines for ΔOED in Fig.2 yields

$$\frac{1}{\sin\left(\frac{n-2}{2n}\pi\right)} = \frac{r}{\sin\left(\frac{n-2}{2n}(\pi-\alpha)\right)} \quad , \quad \text{or}$$

$$r = \frac{\sin\left(\frac{n-2}{2n}(\pi-\alpha)\right)}{\sin\left(\frac{n-2}{2n}\pi\right)} \quad . \tag{5}$$

From (5) and Fig.2 we obtain following equations

$$x = \frac{\sin\left(\frac{n-2}{2n}(\pi-\alpha)\right)}{\sin\left(\frac{n-2}{2n}\pi\right)}\cos\left(\frac{\alpha}{2}\right) \quad , \quad y = \frac{\sin\left(\frac{n-2}{2n}(\pi-\alpha)\right)}{\sin\left(\frac{n-2}{2n}\pi\right)}\sin\left(\frac{\alpha}{2}\right) \quad . \tag{6}$$

Next we show that equations (6) represent a special case of an epitrochoid[3]. An epitrochoid is defined by the following parametric equations

$$x=(a+b)\cos(\varphi)-h\cos((a/b+1)\varphi) \quad , \quad y=(a+b)\sin(\varphi)-h\sin((a/b+1)\varphi) \quad . \tag{7}$$

For the special case $a=1$, and $h=a+b$, from (7) follows

$$x=2(1+b)\sin\left((1+2b)\frac{\varphi}{2b}\right)\sin\left(\frac{\varphi}{2b}\right) \quad , \quad y=-2(1+b)\cos\left((1+2b)\frac{\varphi}{2b}\right)\sin\left(\frac{\varphi}{2b}\right) \quad . \tag{8}$$

Making in (8) the substitution

$$\varphi=\frac{b(\alpha-\pi)}{1+2b} \quad , \tag{9}$$

we obtain

$$x=2(1+b)\sin\left(\frac{\pi-\alpha}{2(1+2b)}\right)\cos\frac{\alpha}{2} \quad , \quad y=2(1+b)\sin\left(\frac{\pi-\alpha}{2(1+2b)}\right)\sin\frac{\alpha}{2} \quad . \tag{10}$$

Comparing (6) and (10), it follows that (6) can be obtained from (10) for $\frac{n-2}{n}=\frac{1}{1+2b}$, or

$$b=\frac{1}{n-2} \quad , \tag{11}$$

and by scaling (10) with the factor

$$s=\frac{1}{2(1+b)\sin\left(\frac{\pi}{2(1+2b)}\right)} \quad . \tag{12}$$

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[3] Eric W. Weisstein. "Epitrochoid." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/Epitrochoid.html