Approximate Angle Trisection and N-Section

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It is well-known that a line can be easily divided into any given number of equal parts, using ruler and compass. However, if we try to do the same with an arbitrary angle, the situation is quite different. For example it has been proven by P. L. Wantzel (1836) using group theory developed by Galois, that it is impossible to trisect every angle using only ruler and compass. However, the search for approximate solutions to this problem turned to be very popular.

In this paper will be shown how we can approximately divide a given angle $0 < \alpha \le \pi$ by a constructible number $n \ge 1$ using a ruler and a compass. It is a trivial task to divide α by 2^k, $k \in N$, and for any other n we have

$$\frac{\alpha}{2^{k+1}} < \frac{\alpha}{n} < \frac{\alpha}{2^k} \quad ,$$

where $2^k < n < 2^{k+1}$.

Let us begin with $n = 3$, and let the given angle α be $\angle AOB$ shown in Fig.1.

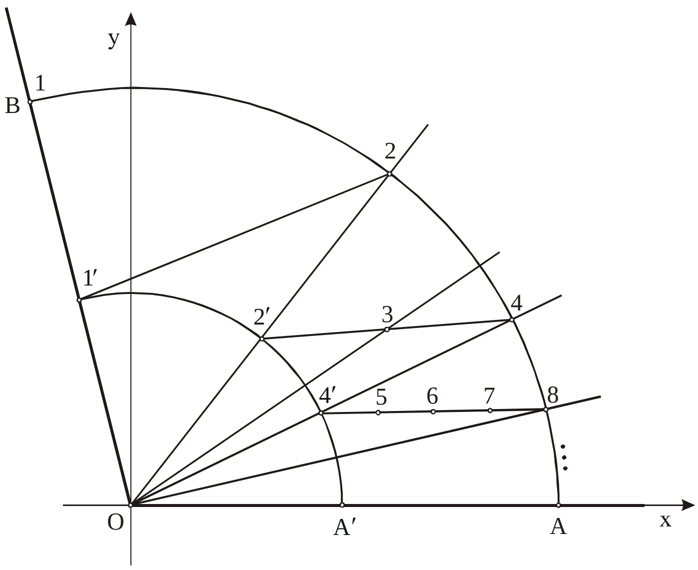

Fig.1

In Fig.1 the following relations hold

$$\angle AO2 = \frac{1}{2}\alpha \,, \quad \angle AO4 = \frac{1}{4}\alpha \,, \quad \angle AO8 = \frac{1}{8}\alpha \,,$$

and

$$\overline{OA'} = \overline{A'A}, \quad \overline{2'3} = \overline{34}, \quad \overline{4'5} = \overline{56} = \overline{67} = \overline{78} \quad .$$

Next we will show that $\angle AO3 \approx \frac{1}{3}\alpha$. Let $\overline{OA'} = r$. From Fig.1 follows

$$\overrightarrow{O3} = \frac{1}{2}\overrightarrow{O2'} + \frac{1}{2}\overrightarrow{O4} \quad ,$$

$$\overrightarrow{O2'} = r\cos\frac{\alpha}{2}\vec{i} + r\sin\frac{\alpha}{2}\vec{j} \quad , \quad \overrightarrow{O4} = 2r\cos\frac{\alpha}{4}\vec{i} + 2r\sin\frac{\alpha}{4}\vec{j} \quad ,$$

$$\overrightarrow{O3} = \frac{r}{2}[(\cos\frac{\alpha}{2} + 2\cos\frac{\alpha}{4})\vec{i} + (\sin\frac{\alpha}{2} + 2\sin\frac{\alpha}{4})\vec{j}] \quad ,$$

and finally

$$\angle AO3 = \arctan\frac{\sin\frac{\alpha}{2} + 2\sin\frac{\alpha}{4}}{\cos\frac{\alpha}{2} + 2\cos\frac{\alpha}{4}} \quad . \tag{1}$$

If we expand the right-hand side of (1) in Taylor series in terms of α , we get

$$\angle AO3 = \frac{1}{3}\alpha - \frac{1}{5184}\alpha^3 - \frac{1}{995328}\alpha^5 + \cdots \quad . \tag{2}$$

From (2) we can conclude that $\angle AO3 \approx \frac{1}{3}\alpha$.

In Fig.2 is shown the relative error for $n = 3$ and $\alpha \in (0, \pi]$.

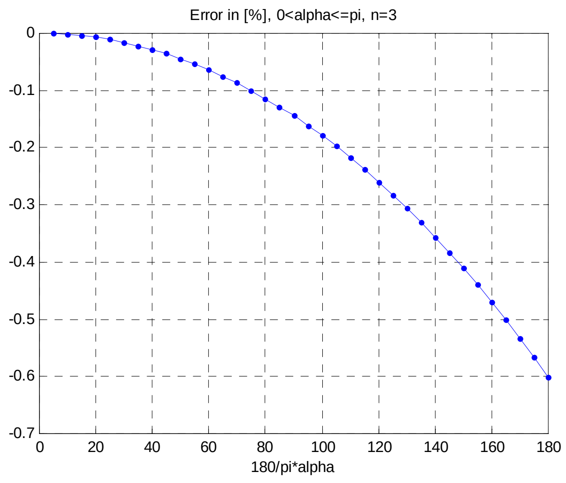

Fig.2

Let us now discuss the general case, Fig.3.

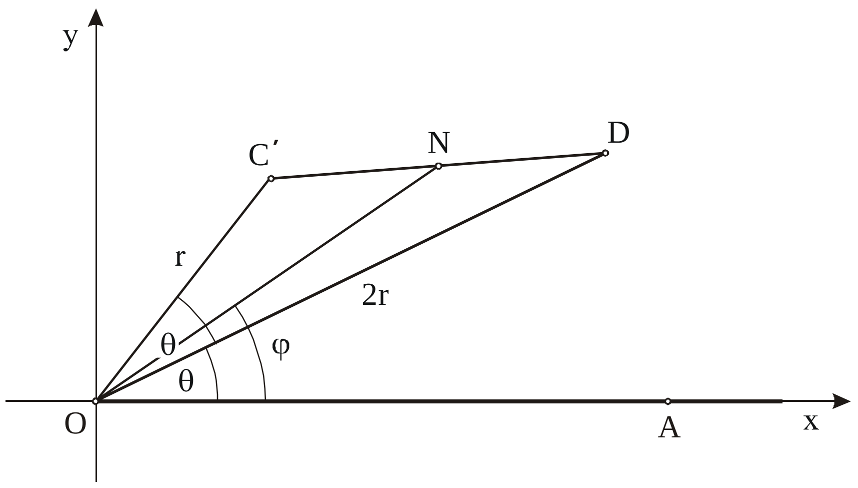

Fig.3

Suppose that $c = 2^k \leq n \leq 2^{k+1} = d$, $\theta = \dfrac{\alpha}{2^{k+1}}$ and $n = 2^k \lambda$, where $1 \leq \lambda \leq 2$, so we have

$$\frac{\alpha}{n} = \frac{2^{k+1}\theta}{2^k \lambda} = \frac{2\theta}{\lambda} \quad .$$

Then with obvious notation we may represent n by the line point N on the line segment

$\overline{C'D}$ in Fig.3 . So we have

$$\frac{\overline{C'N}}{\overline{C'D}} = \frac{n - 2^k}{2^{k+1} - 2^k} = \lambda - 1 \quad \text{, and}$$

$$\begin{aligned}\overrightarrow{ON} &= [1 - (\lambda - 1)]\overrightarrow{OC'} + (\lambda - 1)\overrightarrow{OD} \ , \\ &= (2 - \lambda) r (\cos 2\theta \, \vec{i} + \sin 2\theta \, \vec{j}) + (\lambda - 1) 2r (\cos\theta \, \vec{i} + \sin\theta \, \vec{j}) \ .\end{aligned}$$

Let $\angle AON = \varphi$, then

$$\varphi = \arctan \frac{(2-\lambda)\sin 2\theta + 2(\lambda - 1)\sin\theta}{(2-\lambda)\cos 2\theta + 2(\lambda - 1)\cos\theta} \quad . \qquad (3)$$

From (3) we see that if $\lambda = 1$, then $\varphi = 2\theta = \dfrac{2\theta}{\lambda} = \dfrac{\alpha}{n}$, and likewise if $\lambda = 2$, then $\varphi = \theta = \dfrac{2\theta}{\lambda} = \dfrac{\alpha}{n}$ as expected. For λ such that $2 - \lambda = 2(\lambda - 1)$, it is $\lambda = \dfrac{4}{3}$, from (3) follows

$$\varphi = \arctan\frac{\sin 2\theta + \sin\theta}{\cos 2\theta + \cos\theta} = \arctan\frac{2\sin\frac{3\theta}{2}\cos\frac{\theta}{2}}{2\cos\frac{3\theta}{2}\cos\frac{\theta}{2}} \quad ,$$

so $\varphi = \frac{3\theta}{2}$. In this case $\overrightarrow{ON}$ bisects $\angle DOC'$ and $\varphi = \frac{2\theta}{\lambda} = \frac{\alpha}{n}$. Hence for $\lambda = 1$, $\frac{4}{3}$ and 2, $\varphi = \frac{\alpha}{n}$ and the construction is exact.

It is convenient to expand (3) in Taylor series in terms of $\frac{\alpha}{n}$ for $n \geq 1$, aided by the fact that it must be exact for $\lambda = 1$, $\frac{4}{3}$ and 2. We find

$$\varphi = \frac{\alpha}{n} + \frac{1}{8}(\lambda - 1)(\lambda - \frac{4}{3})(\lambda - 2)\left[K_0(\lambda)\left(\frac{\alpha}{n}\right)^3 - K_2(\lambda)\left(\frac{\alpha}{n}\right)^5 + K_4(\lambda)\left(\frac{\alpha}{n}\right)^7 - \cdots\right] \quad , \qquad (4)$$

where the first few $K_{2i}(\lambda)$ are

$$K_0(\lambda) = 1,$$
$$K_2(\lambda) = \frac{1}{80}(25\lambda^2 - 72\lambda + 48),$$
$$K_4(\lambda) = \frac{1}{13440}(1561\lambda^4 - 9000\lambda^3 + 18960\lambda^2 - 17280\lambda + 5760),$$
$$K_6(\lambda) = \frac{1}{3870720}(181945\lambda^6 - 1573992\lambda^5 + 5585328\lambda^4 - 10402560\lambda^3 + 10725120\lambda^2 - $$
$$- 5806080\lambda + 1290240).$$

It is interesting that $K_2(\lambda), K_4(\lambda), K_6(\lambda)$ have all their roots in the interval $[1, 2]$.

If we write error $R(\alpha, n) = \varphi - \frac{\alpha}{n}$, and relative error

$$\rho = \frac{R(\alpha, n)}{\frac{\alpha}{n}} = \frac{n\varphi}{\alpha} - 1 \quad , \qquad (5)$$

then for $\frac{\alpha}{n} < 1$ from (4) and (5) we get

$$\rho \approx \frac{1}{8}(\lambda - 1)(\lambda - \frac{4}{3})(\lambda - 2)\left(\frac{\alpha}{n}\right)^2\left[K_0(\lambda) - K_2(\lambda)\left(\frac{\alpha}{n}\right)^2\right]. \qquad (6)$$

Then for $\alpha = \pi$ and $1 < n < 2$, numerical search using (3) and (5) gives maximum $|\rho|$ occurring at $n = 1.64$ approximately, with $\rho = -0.03688$ (slightly over 3.5%),and (6) gives $\rho \approx -0.03663$, close to the correct value, suggesting that we have convergence in (4) for $n \geq 1$ and $0 < \alpha \leq \pi$, see Fig.4 and Fig.5.

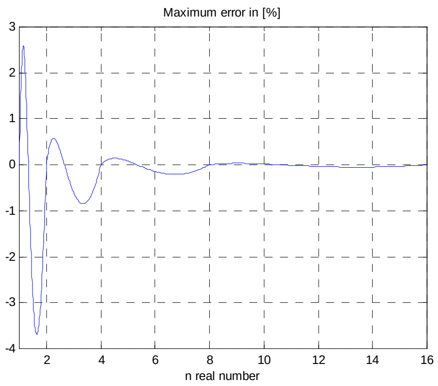

Fig.4

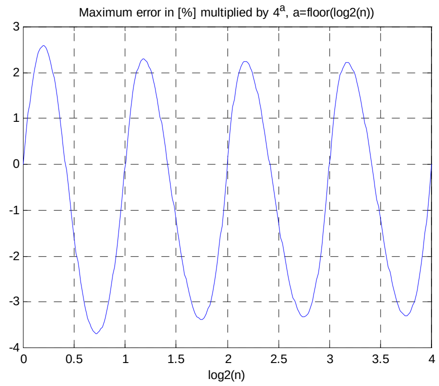

Fig.5

We can achieve more accuracy at the cost of increased constructing effort, see Fig.6.

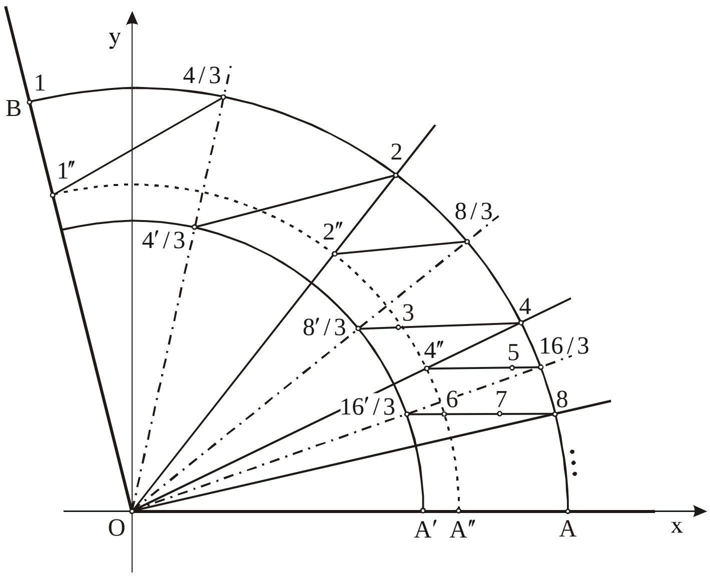

Fig.6

In Fig.6 the following relations hold

$$\angle AO4/3 = \frac{3}{4}\alpha, \ \angle AO8/3 = \frac{3}{8}\alpha, \ \angle AO16/3 = \frac{3}{16}\alpha,$$

$$\overline{OA'} = \frac{2}{3}\overline{OA}, \ \overline{OA''} = \frac{3}{4}\overline{OA},$$

$$\overline{8'/33} = \frac{1}{4}\overline{8'/34}, \; \overline{516/3} = \frac{1}{4}\overline{4''16/3},$$

$$\overline{16'/36} = \frac{1}{4}\overline{16'/38}, \; \overline{67} = \overline{78} = \frac{3}{8}\overline{16'/38} \quad .$$

According to Fig.6 there are two cases:

If $2^k \le n < \frac{4}{3}2^k$, then

$$\varphi = \arctan \frac{(4\cdot 2^k - 3n)\sin\frac{\alpha}{2^k} + 4(n-2^k)\sin\frac{3\alpha}{4\cdot 2^k}}{(4\cdot 2^k - 3n)\cos\frac{\alpha}{2^k} + 4(n-2^k)\cos\frac{3\alpha}{4\cdot 2^k}} \, ,$$

else if $\frac{4}{3}2^k \le n < 2^{k+1}$, then

$$\varphi = \arctan \frac{2(2^{k+1} - n)\sin\frac{3\alpha}{4\cdot 2^k} + (3n - 4\cdot 2^k)\sin\frac{\alpha}{2^{k+1}}}{2(2^{k+1} - n)\cos\frac{3\alpha}{4\cdot 2^k} + (3n - 4\cdot 2^k)\cos\frac{\alpha}{2^{k+1}}} \, .$$

The relative error is shown in Fig.7 and Fig.8.

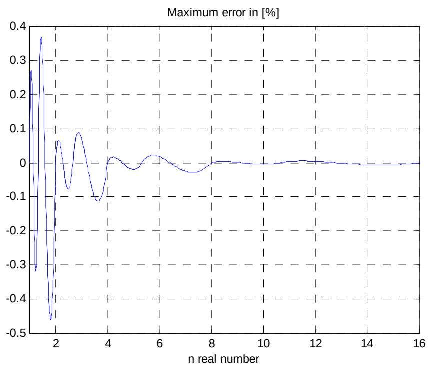

Fig.7

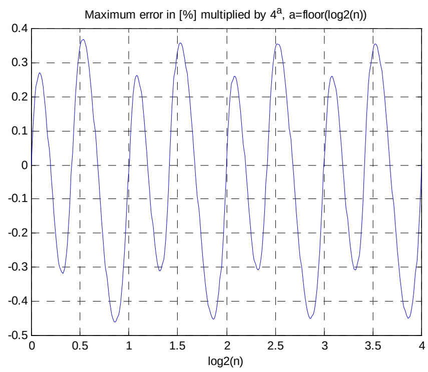

Fig.8