All points lead to the symmedian point[1]

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Let ABC be an arbitrary triangle and P = (u : v : w) an arbitray point, as shown in Fig. 1.

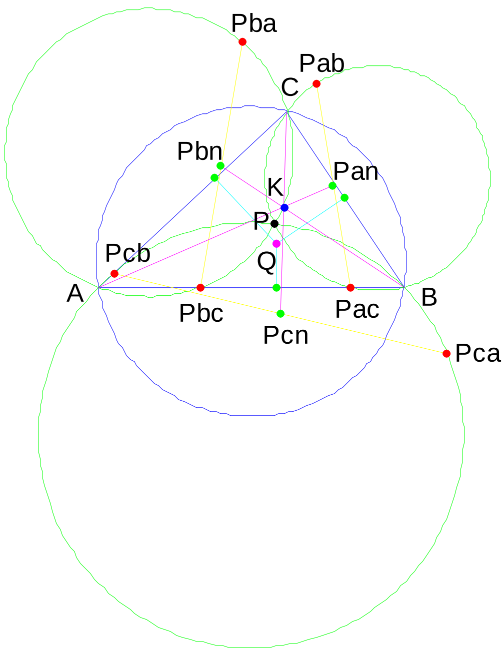

Fig. 1

Let Pab, Pac be the intersection points of circle Circ(P,B,C) with lines CA, AB respectively. Let Pbc, Pba be the intersection points of Circ(P,C,A) with lines AB, BC respectively. Let Pca, Pcb be the intersection points of Circ(P,A,B) with lines BC, CA respectively. The barycentric coordinates of points Pab, Pac, Pbc, Pba, Pca, Pcb are as follows

$$P_{ab} = (b^2 + p : 0 : -p) \quad , \quad P_{ac} = (c^2 + p : -p : 0)$$
$$P_{bc} = (-q : c^2 + q : 0) \quad , \quad P_{ba} = (0 : a^2 + q : -q)$$
$$P_{ca} = (0 : -r : a^2 + r) \quad , \quad P_{cb} = (-r : 0 : b^2 + r)$$

where $p = t/u, q = t/v, r = t/w$, and $t = -(a^2 \cdot v \cdot w + b^2 \cdot w \cdot u + c^2 \cdot u \cdot v)/(u+v+w)$.

Let Pan, Pbn, Pcn be the middle points of segments (Pab, Pac), (Pbc, Pba), (Pca, Pcb) respectively. Then lines APan, BPbn, CPcn are concurrent at the symmedian point[2] X(6), or

[1] Symmedian Point: http://mathworld.wolfram.com/SymmedianPoint.html

$$K = (a^2 : b^2 : c^2) \ ,$$

which is independent of the arbitrary point P.

Let Pam, Pbm, Pcm be the middle points of segments (Pba, Pca), (Pcb, Pab), (Pac, Pbc) respectively. Then lines through Pam, Pbm, Pcm and perpendicular respectively to the lines BC, CA, AB are concurrent at point Q = (x : y : z), where

$$x = (S_a + 2 \cdot p) \cdot a^2 - S_b \cdot r - S_c \cdot q \ ,$$

and $S_a = b^2 + c^2 - a^2$, $S_b = c^2 + a^2 - b^2$, $S_c = a^2 + b^2 - c^2$.

For P = I, that means the incenter of triangle ABC, we obtain Q = I. For P = G, the centroid of triangle ABC, we obtain Q = O, the circumcenter of triangle ABC. For P = H, the orthocenter of triangle ABC, Q = X(382).

[2] Three half-circles: http://mathforum.org/kb/message.jspa?messageID=7563345