Three Concurrent Lines

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Let L1, Ly, L2 be three planar lines that intersect each other at the same point P. Let Lx be a line perpendicular to Ly that intersects L1, Ly, and L2 at points A, O, and C respectively. Two rays emanating from O at an angle ‘g’ with respect to Lx, and symmetrically to Ly, intersect L1 and L2 at points B and D respectively, see Fig. 1.

1. Then lines L(C,B) and L(A,D) intersect each other at a point E that lies on Ly.

Moreover let B1 and D1 be the intersection points of L1 and L2 respectively with two other rays from O at an angle ‘g1’ with respect to Lx.

2. Then lines L(D1,B) and L(B1,D) also intersect each other at a point F that lies on Ly.

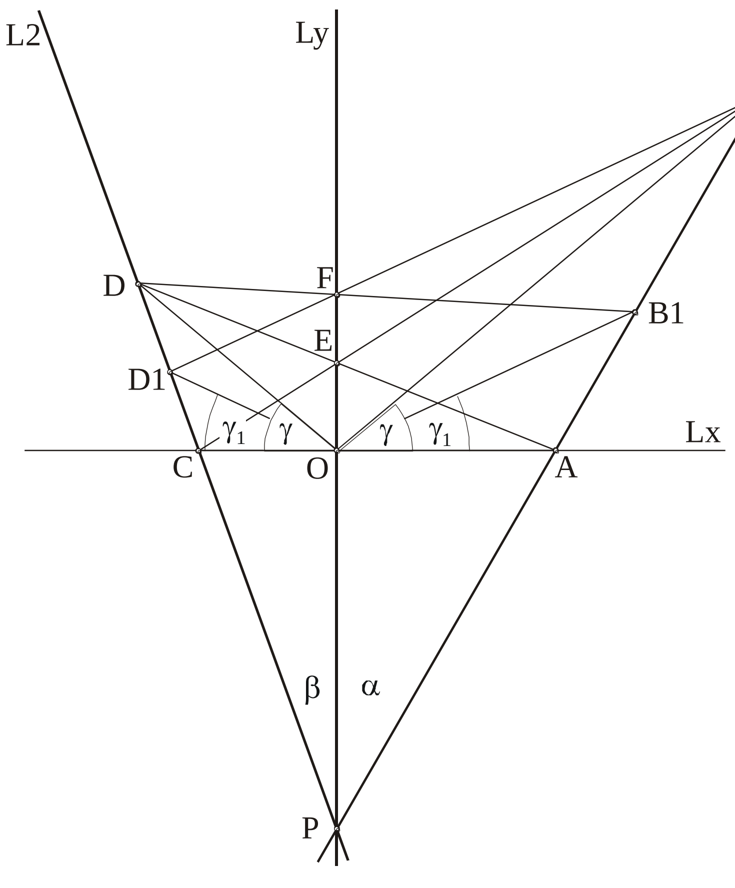

Fig. 1

Proof. For the sake of simplicity let Lx and Ly be respectively the x- and y-axis of a rectangular coordinate system, and point P has coordinates [0,-1]. Let ‘a’ be the angle between L1 and Ly, and ‘b’ the angle between Ly and L2. We have

$$A = [\tan(a), 0] ,$$
$$B = 1/(1-\tan(a)*\tan(g))*[\tan(a), \tan(a)*\tan(g)] ,$$
$$B1= 1/(1-\tan(a)*\tan(g1))*[\tan(a), \tan(a)*\tan(g1)] ,$$

$$C = [-\tan(b), 0] ,$$
$$D = 1/(1-\tan(b)*\tan(g))*[-\tan(b), \tan(b)*\tan(g)] ,$$
$$D1= 1/(1-\tan(b)*\tan(g1))*[-\tan(b), \tan(b)*\tan(g1)] .$$

The line L(C,B) intersects Ly at a point E with the ordinate

$$Ey = (By-Cy)/(Bx-Cx)*(-Cx)+Cy , \quad \text{or}$$

$$Ey = \tan(a)*\tan(b)*\tan(g)/(\tan(a)+\tan(b)-\tan(a)*\tan(b)*\tan(g)) .$$

The same expression for Ey we obtain for the line L(A,D). This proves the first statement.

The line L(D1,B) intersects Ly at a point F with the ordinate

$$Fy = (By-D1y)/(Bx-D1x)*(-D1x)+D1y , \quad \text{or}$$

$$Fy = \tan(a)*\tan(b)*(\tan(g)+\tan(g1))/(\tan(a)+\tan(b)-\tan(a)*\tan(b)*(\tan(g)+\tan(g1))) .$$

The same expression for Fy we obtain for the line L(B1,D). This proves the second statement.

Let Q be the intersection point of line L(D,B) and Lx, then the abscise of Q is

$$Qx = -Dy*(Bx-Dx)/(By-Dy)+Dx , \quad \text{or}$$

$$Qx = 2*\tan(a)*\tan(b)/(\tan(b)-\tan(a)) .$$

We observe that Qx is independent of ray angle ‘g’. This means that Q is a perspectivity point and points (P,A,B1,B) and (P,C,D1,D) have the same cross ratio. This cross ratio is

$$CR = PB1*AB/(PB*AB1) , \quad \text{or}$$

$$CR = \tan(g)/\tan(g1) .$$

We observe that CR does not depend on the angles ‘a’ and ‘b’.