Miscellaneous Results on Tetrahedron

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We start with Ceva's theorem for tetrahedron in barycentric coordinates[1],

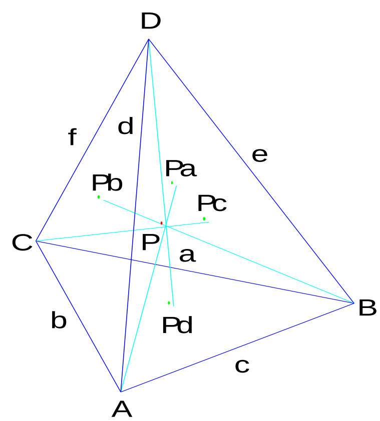

Fig. 1

Pa(D)*Pb(C)*Pc(B)*Pd(A) : Pa(B)*Pb(A)*Pc(D)*Pd(C) : Pa(C)*Pb(D)*Pc(A)*Pd(B) = 1 : 1 : 1 , (1)

where P is an arbitrary point and Pa = (0 : Pa(B) : Pa(C) : Pa(D)) , Pb = (Pb(A) : 0 : Pb(C) : Pb(D)) , Pc = (Pc(A) : Pc(B) : 0 : Pc(D)) , Pd = (Pd(A) : Pd(B) : Pd(C) : 0) are its traces on the respective faces of tetrahedron ABCD, see Fig. 1.

From relation (1) it follows immediately that cevians to the points Pai, Pbi, Pci, Pdi built as isotomic conjugates of traces Pa, Pb, Pc, Pd, are also concurrent.

Now let us consider the insphere and exspheres of tetrahedron. Let Ged and Nad be respectively the points where the insphere and the respective exsphere meet the tetrahedron face ABC. Point Ged is isogonal conjugate to Nad. Let α, β, γ be the angles between the tetrahedron faces sharing the sides a, b, c respectively. The barycentric coordinates of Nad and Ged are as follows

$$\text{Nad} = (a\cdot\tan(\alpha/2) : b\cdot\tan(\beta/2) : c\cdot\tan(\gamma/2) : 0) , \quad (2)$$

$$\text{Ged} = (a/\tan(\alpha/2) : b/\tan(\beta/2) : c/\tan(\gamma/2) : 0) , \quad (3)$$

[1] A Generalization of Ceva's Theorem for Tetrahedron: http://tetraceva.webs.com/

or

$$Ged = (a^2/(a*\tan(\alpha/2)) : b^2/(b*\tan(\beta/2)) : c^2/(c*\tan(\gamma /2)) : 0) . \quad (4)$$

Comparing (4) and (2) follows that Nad and Ged are isogonal conjugate to each other. From the spherical trigonometry we have the nice formula

$$\tan(\alpha/2) = sqrt(\sin(s-bb)*\sin(s-cc)/(\sin(s)*\sin(s-aa))) , \quad (5)$$

where aa is the angle between sides c, e; angle bb is between sides e, a; angle cc is between sides a, c, and s = (aa+bb+cc)/2 . Similarly we find tan(β/2) and tan(γ /2).

The analogy to Dandelin spheres[2] is obvious, so the points Ged and Nad are foci of the ellipse which passes through the reflection points of the rays from Ged to Nad on the sides of triangle ABC. Motivated by this property we define the following mapping.

Let P = (u : v : w) be an arbitrary point and Q its isogonal conjugate with respect to triangle ABC. Let Qa, Qb, Qc be the mirror points of Q with respect to the sides a, b, c of triangle ABC. Let Ra, Rb, Rc be the intersection points of lines PQa, PQb, PQc with triangle sides BC, CA, AB respectively. Now lines ARa, BRb, CRc are concurrent at point T with barycentric coordinates (Tx:Ty:Tz),

$$Tx = 1/(u*(w*(b^2*(w+v)+v*(c^2-a^2))+c^2*v^2)) ,$$

$$Ty = 1/(v*(u*(c^2*(u+w)+w*(a^2-b^2))+a^2*w^2)) ,$$

$$Tz = 1/(w*(v*(a^2*(v+u)+u*(b^2-c^2))+b^2*u^2)) .$$

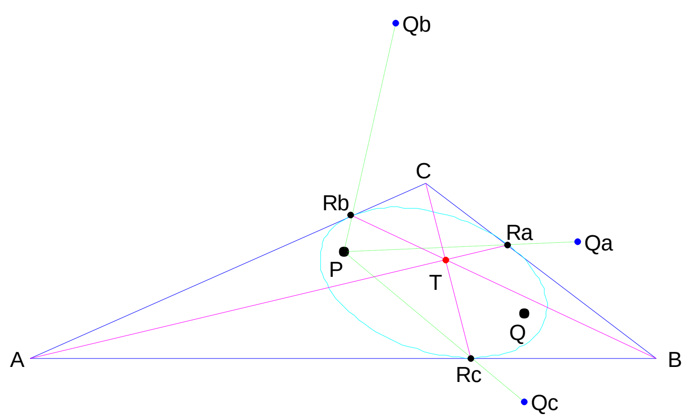

Fig. 2

[2] Dandelin Spheres: http://mathworld.wolfram.com/DandelinSpheres.html

For P = (a : b : c) , that is P = X(1) the incenter of triangle ABC, center T represents the Gergonne point X(7). For P = (1 : 1 : 1) , that is P = X(2) the centroid of triangle ABC, T represents X(598). The following relation

$$PRa+RaQ = PRb+RbQ = PRc+RcQ \quad , \qquad (6)$$

also holds for arbitrary P .

Let P = (x : y : z : w) be an arbitrary point, then Q = (Sa^2/x : Sb^2/y : Sc^2/z : Sd^2/w) is the isogonal conjugate of P, where Sa, Sb, Sc, Sd are the areas of tetrahedron faces. Let Ra, Rb, Rc, Rd be the reflection points of the rays from P to Q on the respective faces of tetrahedron, then a relation similar to (6) holds

$$PRa+RaQ = PRb+RbQ = PRc+RcQ = PRd+RdQ \quad .$$

Let P = (px : py : pz : pw) and Q = (qx : qy : qz : qw) be arbitrary points, then we call point S = (px^2/qx : py^2/qy : pz^2/qz : pw^2/qw) the iso-P conjugate of Q. This mapping represents a generalization of the isotomic and isogonal conjugate mapping. The geometric interpretation of this generalized mapping will be explained by means of the respective cevian traces on face ABC of tetrahedron, see Fig. 3.

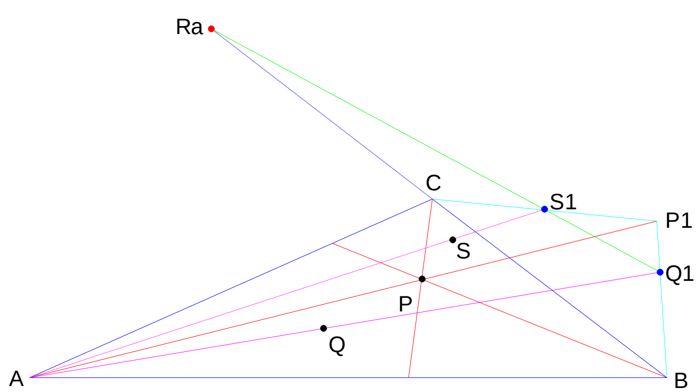

Fig.3

In this planar case for the traces of the points on face ABC, for example P, instead of (px : py : pz : 0) we use the notation (px : py : pz).
Let P1 be an arbitrary point on the extension of line AP. Let Q1 and S1 be the intersection points of lines AP and AS with the lines BP1 and CP1 respectively. Then the line Q1S1 intersects the line BC at the point Ra = (0 : py : -pz), which depends only on the coordinates of P. This gives us the possibility to

construct point S1 for given P and Q. It is obvious that line AS1 coincides with the cevian of S from A. Similarly we obtain points Rb = (-px : 0 : pz) and Rc = (px : -py : 0) for the respective edges b and c of triangle ABC. We note that Ra, Rb, Rc are collinear, that is

$$\det([0\ \ py\ \ -pz\ ;\ -px\ \ 0\ \ pz\ ;\ px\ \ -py\ 0\ ]) = 0\ ,$$

and they lie on the tripolar line[3] corresponding to P.

Let us now turn to the incenter It of tetrahedron. Center It is the intersection point of planes that bisect the angles between the tetrahedron faces. Let Sa, Sb, Sc, Sd be the areas of tetrahedron faces, then It has barycentric coordinates

$$It = (\ Sa : Sb : Sc : Sd\ )\ . \qquad (7)$$

The trace of It in the face ABC has barycentric coordinates

$$Itd = (\ Sa : Sb : Sc : 0\ )\ . \qquad (8)$$

Relation (8) represents a generalization of angle bisector theorem for tetrahedron.

Let EtA, EtB, EtC, EtD be the excenters of tetrahedron corresponding to the tetrahedron vertices A, B, C, D respectively. We show that A, B, EtA, EtB are complanar. Since A= (1 : 0 : 0 : 0), B=(0 : 1 : 0 : 0), EtA = (-Sa : Sb : Sc : Sd), EtB = (Sa : –Sb : Sc : Sd), it follows

$$\det([1\ 0\ 0\ 0\ ;\ 0\ 1\ 0\ 0\ ;\ -Sa\ Sb\ Sc\ Sd\ ;\ Sa\ -Sb\ Sc\ Sd]) = 0\ .$$

Also C, D, EtA, EtB are complanar. Since C= (0 : 0 : 1 : 0), D=(0 : 0 : 0 : 1), it follows

$$\det([0\ 0\ 1\ 0\ ;\ 0\ 0\ 0\ 1\ ;\ -Sa\ Sb\ Sc\ Sd\ ;\ Sa\ -Sb\ Sc\ Sd]) = 0\ .$$

The same holds for other edges of tetrahedron. We define the following planes:
E1 = {B, D, EtA, EtC}, E2 = {C, D, EtA, EtB}, E3 = {A, D, EtB, EtC}, E4 = {A, C, EtB, EtD},
E5 = {A, B, EtC, EtD}, E6 = {B, C, EtA, EtD}. The planes E1, E2, E4, E5 are concurrent at point Ef1, planes E1, E3, E4, E6 are concurrent at point Ef2, planes E2, E3, E5, E6 are concurrent at point Ef3, where Ef1 = (-Sa : Sb : Sc : -Sd), Ef2 = (-Sa : -Sb : Sc : Sd), Ef3 = (-Sa : Sb : -Sc : Sd). Points Ef1, Ef2, Ef3 are exactly the far excenters of tetrahedron ABCD. The far excenters are the centers of spheres that touch the planar extensions of the faces of tetrahedron ABCD. Centers Ef1, Ef2, Ef3 lie on the tetrapolar plane E4i of the incenter It = (Sa : Sb : Sc : Sd) of tetrahedron ABCD, where

$$E4i:\ \ x/Sa + y/Sb + z/Sc + w/Sd = 0\ .$$

Let r, ra, rb, rc, rd be radii of insphere and exspheres of tetrahedron ABCD, then we have following relations

[3] Tripolar line: http://www.paideiaschool.org/teacherpages/steve_sigur/resources/ultimate%20triangle/ultimte.html

$$3*V/r = Sa+Sb+Sc+Sd \;, \tag{9}$$
$$3*V/ra = -Sa+Sb+Sc+Sd \;, \tag{10}$$
$$3*V/rb = Sa -Sb+Sc+Sd \;, \tag{11}$$
$$3*V/rc = Sa+Sb -Sc+Sd \;, \tag{12}$$
$$3*V/rd = Sa+Sb+Sc -Sd \;, \tag{13}$$

where V is the volume of tetrahedron. From (9), (10), (11), (12), (13) we obtain

$$2/r = 1/ra + 1/rb + 1/rc + 1/rd \;. \tag{14}$$

The circumcenter Ot or the center of the circumsphere of tetrahedron is the intersection point of planes through midpoints of sides and orthogonal to those sides of tetrahedron. Let a, b, c, d, e, f be the side lengths of tetrahedron, then Ot has barycentric coordinates (OtA : OtB : OtC : OtD) ,

$$OtA = d\^{}2*a\^{}2*(f\^{}2+e\^{}2-a\^{}2) + b\^{}2*e\^{}2*(a\^{}2+f\^{}2-e\^{}2) + c\^{}2*f\^{}2*(e\^{}2+a\^{}2-f\^{}2) - 2*a\^{}2*e\^{}2*f\^{}2 \;, \tag{15}$$

$$OtB = e\^{}2*b\^{}2*(f\^{}2+d\^{}2-b\^{}2) + c\^{}2*f\^{}2*(d\^{}2+b\^{}2-f\^{}2) + a\^{}2*d\^{}2*(b\^{}2+f\^{}2-d\^{}2) - 2*b\^{}2*d\^{}2*f\^{}2 \;, \tag{16}$$

$$OtC = f\^{}2*c\^{}2*(e\^{}2+d\^{}2-c\^{}2) + b\^{}2*e\^{}2*(d\^{}2+c\^{}2-e\^{}2) + a\^{}2*d\^{}2*(c\^{}2+e\^{}2-d\^{}2) - 2*c\^{}2*e\^{}2*d\^{}2 \;, \tag{17}$$

$$OtD = d\^{}2*a\^{}2*(b\^{}2+c\^{}2-a\^{}2) + e\^{}2*b\^{}2*(c\^{}2+a\^{}2-b\^{}2) + f\^{}2*c\^{}2*(a\^{}2+b\^{}2-c\^{}2) - 2*a\^{}2*b\^{}2*c\^{}2 \;. \tag{18}$$

From (15), (16), (17), (18) we obtain

$$OtA + OtB + OtC + OtD = 288*V\^{}2 \;, \tag{19}$$

where V is the volume of tetrahedron. Relation (19) is similar to the sum of barycentric coordinates of the circumcenter of triangle ABC:

$$a\^{}2*(b\^{}2+c\^{}2-a\^{}2) + b\^{}2*(c\^{}2+a\^{}2-b\^{}2) + c\^{}2*(a\^{}2+b\^{}2-c\^{}2) = 16*S\^{}2, \tag{20}$$

where S is the area of triangle ABC.

We recall the Cayley-Menger determinant[4] for the volume of tetrahedron

$$288*V\^{}2 = |\mathbf{K}| \;, \tag{21}$$

where

$$\mathbf{K} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & d_{12}^2 & d_{13}^2 & d_{14}^2 \\ 1 & d_{21}^2 & 0 & d_{23}^2 & d_{24}^2 \\ 1 & d_{31}^2 & d_{32}^2 & 0 & d_{34}^2 \\ 1 & d_{41}^2 & d_{42}^2 & d_{43}^2 & 0 \end{bmatrix}, \tag{22}$$

[4] Cayley-Menger determinant: http://mathworld.wolfram.com/Cayley-MengerDeterminant.html

and

$$d_{23} = d_{32} = a, d_{13} = d_{31} = b, d_{12} = d_{21} = c, d_{14} = d_{41} = d, d_{24} = d_{42} = e, d_{34} = d_{43} = f.$$

Comparing (19) and (21) we obtain

$$OtA = K_{12}, \; OtB = K_{13}, \; OtC = K_{14}, \; OtD = K_{15} \;, \tag{23}$$

where K_{ij} are the respective cofactors of matrix **K**.
In analogy to the Euler Line of an arbitrary triangle, in Fig. 4 is shown the Euler Plane (green area) of an arbitrary tetrahedron ABCD.

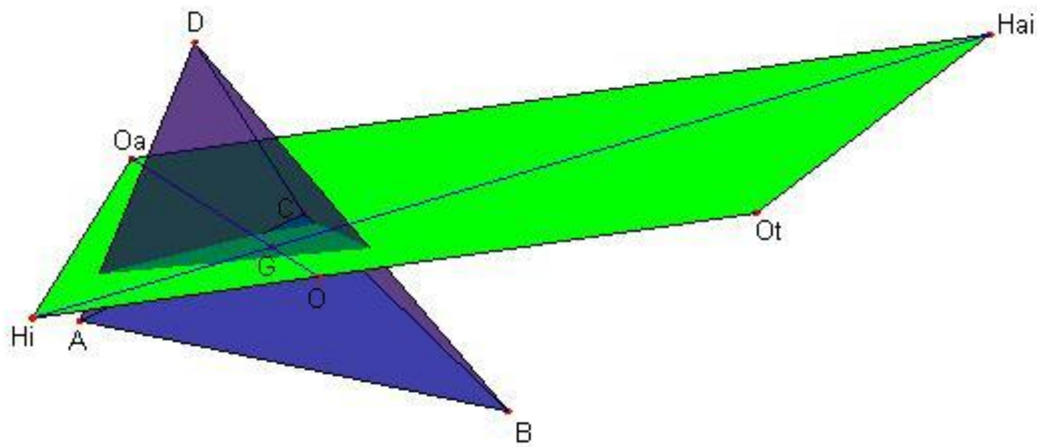

Fig. 4

In Fig. 4, points O, G are the circumcenter and the centroid of tetrahedron ABCD respectively. Point Hi is the isogonal conjugate of O, in analogy to the triangle, where the orthocenter is the isogonal conjugate of circumcenter. It is well known that an arbitrary tetrahedron has no orthocenter, and in the case of an orthocentric tetrahedron the orthocenter H does not coincide with point Hi, but it lies on the line OG of the Euler Plane, where the well known relation OG = GH holds.
Point Ot is the circumcenter of the tangential tetrahedron, in analogy to the tangential triangle[5]. The circumcenter of the tangential triangle of an arbitrary triangle ABC, lies on the Euler Line of triangle ABC, point X(26).
Point Oa is the circumcenter of the anticomplementary tetrahedron, in analogy to the anticomplementary triangle[6]. Point Hai is the isogonal conjugate of Oa. In analogy to the triangle, point Hai corresponds to the de Longchamps point X(20), which lies on the Euler Line of the triangle.
For the points O, G, Oa and Hi, G, Hai holds

$$3*OG = GOa \;, \; 3*HiG = GHai \;. \tag{24}$$

Consequently, lines OHi and OaHai are parallel.

[5] Tangential triangle: http://mathworld.wolfram.com/TangentialTriangle.html
[6] Anticomplementary triangle: http://mathworld.wolfram.com/AnticomplementaryTriangle.html