Incenter and Centroid

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Let ABC be an arbitrary triangle as shown in Fig. 1.

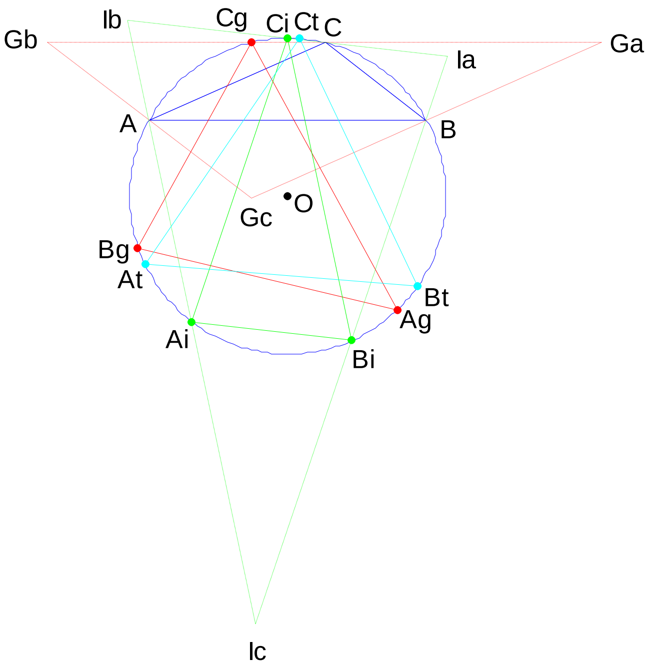

Fig. 1

Triangles IaIbIc and GaGbGc are respectively the excentral and anticomplementary triangle of triangle ABC, or in other words they are respectively the precevian triangles of incenter I and centroid G of triangle ABC.

The circumcircle of triangle ABC is the nine-point circle[1] of both triangles IaIbIc and GaGbGc.

Points Ai, Bi, Ci are the intersections of respective sides of triangle IaIbIc with the circumcircle of triangle ABC. The Simson lines corresponding to Ai, Bi, Ci intersect at the Spieker Point of triangle ABC. According to the Pascal's Theorem let P1, P2, P3 be the intersection points of the line pairs {ABi, AiB}, {BCi, BiC}, {CAi, CiA} respectively. Points P1, P2, P3

[1] Nine-point circle: https://en.wikipedia.org/wiki/Nine-point_circle

are collinear and lie exactly on the line OI of triangle ABC, where O is the circumcenter of triangle ABC, or in other words they lie on the Euler line of triangle IaIbIc.

Points Ag, Bg, Cg are the intersections of respective sides of triangle GaGbGc with the circumcircle of triangle ABC. The Simson lines corresponding to Ag, Bg, Cg are concurrent[2]. According to the Pascal's Theorem let Q1, Q2, Q3 be the intersection points of the line pairs {ABg, AgB}, {BCg, BgC}, {CAg, CgA} respectively. Points Q1, Q2, Q3 are collinear and lie exactly on the Euler line of triangle ABC[3], or in other words they lie on the Euler line of triangle GaGbGc.

We can continue the process by taking triangles AiBiCi, AgBgCg as starting triangles to construct their respective excentral and anticomplementary triagles. After n iterations we have the barycentric coordinates

$$\begin{aligned} A_i^{n+1} &= [a_n^2 : b_n \cdot (c_n - b_n) : c_n \cdot (b_n - c_n)], \\ B_i^{n+1} &= [a_n \cdot (c_n - a_n) : b_n^2 : c_n \cdot (a_n - c_n)], \\ C_i^{n+1} &= [a_n \cdot (b_n - a_n) : b_n \cdot (a_n - b_n) : c_n^2], \end{aligned} \tag{1}$$

with respect to triangle $A_i^n B_i^n C_i^n$, where

$$\begin{aligned} a_n &= 2 \cdot R \cdot \sin\left(\frac{2^n - (-1)^n}{3 \cdot 2^n} \cdot \pi + (-1)^n \cdot \frac{\alpha}{2^n}\right), \\ b_n &= 2 \cdot R \cdot \sin\left(\frac{2^n - (-1)^n}{3 \cdot 2^n} \cdot \pi + (-1)^n \cdot \frac{\beta}{2^n}\right), \\ c_n &= 2 \cdot R \cdot \sin\left(\frac{2^n - (-1)^n}{3 \cdot 2^n} \cdot \pi + (-1)^n \cdot \frac{\gamma}{2^n}\right), \end{aligned} \tag{2}$$

where R is the radius of the circumcircle of triangle ABC, and the barycentric coordinates

$$\begin{aligned} A_g^{n+1} &= [a_n^2 : c_n^2 - b_n^2 : b_n^2 - c_n^2], \\ B_g^{n+1} &= [c_n^2 - a_n^2 : b_n^2 : a_n^2 - c_n^2], \\ C_g^{n+1} &= [b_n^2 - a_n^2 : a_n^2 - b_n^2 : c_n^2], \end{aligned} \tag{3}$$

with respect to triangle $A_g^n B_g^n C_g^n$, where

$$\begin{aligned} a_n &= 2 \cdot R \cdot \sin(2^n \cdot \alpha), \\ b_n &= 2 \cdot R \cdot \sin(2^n \cdot \beta), \\ c_n &= 2 \cdot R \cdot \sin(2^n \cdot \gamma). \end{aligned} \tag{4}$$

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[2] Some results on side-bisector reflected triangle: http://trisectlimacon.webs.com/side_bisector_triangle1.pdf

[3] Circle Connections: http://trisectlimacon.webs.com/Circle_Connections.pdf

For n = 0 ,

$a_0 = a, b_0 = b, c_0 = c,$ and $A_i^1 = A_i, B_i^1 = B_i, C_i^1 = C_i$, and $A_g^1 = A_g, B_g^1 = B_g, C_g^1 = C_g$,

and triangle $A_i^0B_i^0C_i^0 = A_g^0B_g^0C_g^0 = ABC$.

Displacement angles of triangle vertices $A_i^{n+1}, B_i^{n+1}, C_i^{n+1}$, with O as rotation center (positive displacements counter clockwise) are

$$\begin{aligned} \Delta\alpha_i^{n+1} &= \gamma_i^n - \beta_i^n \ , \\ \Delta\beta_i^{n+1} &= \alpha_i^n - \gamma_i^n \ , \\ \Delta\gamma_i^{n+1} &= \beta_i^n - \alpha_i^n \ , \end{aligned} \tag{5}$$

where $\alpha_i^n, \beta_i^n, \gamma_i^n$,are respective angles of triangle $A_i^nB_i^nC_i^n$, and for vertices $A_g^{n+1}, B_g^{n+1}, C_g^{n+1}$

$$\begin{aligned} \Delta\alpha_g^{n+1} &= 2\cdot(\gamma_g^n - \beta_g^n)\ , \\ \Delta\beta_g^{n+1} &= 2\cdot(\alpha_g^n - \gamma_g^n)\ , \\ \Delta\gamma_g^{n+1} &= 2\cdot(\beta_g^n - \alpha_g^n)\ , \end{aligned} \tag{6}$$

where $\alpha_g^n, \beta_g^n, \gamma_g^n$, are respective angles of triangle $A_g^nB_g^nC_g^n$.

For n = 0,

$\alpha_i^0 = \alpha_g^0 = \alpha,\ \beta_i^0 = \beta_g^0 = \beta,\ \gamma_i^0 = \gamma_g^0 = \gamma$, and triangle $A_i^0B_i^0C_i^0 = A_g^0B_g^0C_g^0 = ABC$.

From (5) and (2) we have the total displacement angles

$$\begin{aligned} \Delta\alpha_t &= \sum_{n=0}^{\infty}\Delta\alpha_i^{n+1} = \frac{2}{3}\cdot(\gamma - \beta)\ , \\ \Delta\beta_t &= \sum_{n=0}^{\infty}\Delta\beta_i^{n+1} = \frac{2}{3}\cdot(\alpha - \gamma)\ , \\ \Delta\gamma_t &= \sum_{n=0}^{\infty}\Delta\gamma_i^{n+1} = \frac{2}{3}\cdot(\beta - \alpha)\ . \end{aligned} \tag{7}$$

In the limit for $n \to \infty$, from (2) we also have

$$a_n = b_n = c_n = \sqrt{3}\cdot R\ , \tag{8}$$

that means we obtain an equilateral triangle, that in fact is the circumtangential triangle[4] AtBtCt of triangle ABC, shown in Fig. 1. It can be constructed by rotating vertices A, B, C about O by respective angles in (7).

[4] Circumtangential Triangle: http://mathworld.wolfram.com/CircumtangentialTriangle.html

It is well-known that circumcircle is the isogonal conjugate of the line at infinity, which has the equation

$$x + y + z = 0\,. \qquad (9)$$

The circumcircle also is the isotomic conjugate of the line L shown in Fig. 2.

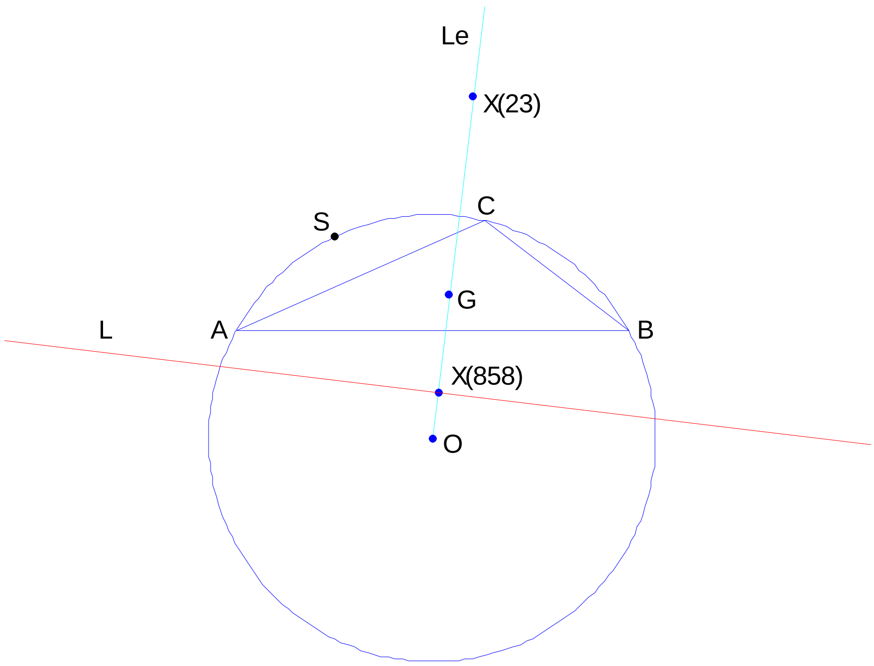

Fig. 2

Line L has the equation

$$a^2x + b^2y + c^2z = 0\,. \qquad (10)$$

It is orthogonal to the Euler line (Le) and passes through center X(858), which is complement of center X(23), and X(23) is the inverse point of G with respect to the circumcircle. The following relation holds

$$2 \cdot \overline{X(858)G} = \overline{GX(23)}\,. \qquad (11)$$

The isotomic conjugate of the point at infinity that corresponds to the line (10), is the Steiner Point[5], or X(99).

[5] Steiner Point: http://mathworld.wolfram.com/SteinerCircumellipse.html

The coefficients of equations (9) and (10) are respectively the squared coordinates of centroid G = (1:1:1) and incenter I = (a : b : c) of triangle ABC. This result can be generalized according to the generalization of the isotomic and isogonal conjugate mapping[6].

Let $P=(p_x:p_y:p_z)$, $Q=(q_x:q_y:q_z)$ be two arbitrary points. Then the iso-P conjugate of line

$$q_x^2\cdot x+q_y^2\cdot y+q_z^2\cdot z=0\ , \tag{12}$$

is the same circumconic as the iso-Q conjugate of line

$$p_x^2\cdot x+p_y^2\cdot y+p_z^2\cdot z=0\ , \tag{13}$$

and its equation is

$$p_x^2q_x^2yz+p_y^2q_y^2zx+p_z^2q_z^2xy=0\ . \tag{14}$$

To prove (14) we substitute the equation for the absolute coordinates $z=1-x-y$ in (12) and for $y=0, y=1$, we obtain two points $Q_1=(q_z^2:0:-q_x^2)$ and $Q_2=(q_y^2:q_z^2-q_x^2:-q_y^2)$ of line (12). For an arbitrary point of line (12) we have $Q_a=(1-t)\cdot Q_1+t\cdot Q_2$, or

$$Q_a=(\ t\cdot(q_y^2-q_z^2)+q_z^2\ :\ t\cdot(q_z^2-q_x^2)\ :\ t\cdot(q_x^2-q_y^2)-q_x^2\ )\ , \tag{15}$$

where t is a real parameter and $t\in(-\infty,\infty)$. For a point $U=(x:y:z)$ that is the iso-P conjugate of (15) we have

$$x=\frac{p_x^2}{t\cdot(q_y^2-q_z^2)+q_z^2}\ ,\quad y=\frac{p_y^2}{t\cdot(q_z^2-q_x^2)}\ ,\quad z=\frac{p_z^2}{t\cdot(q_x^2-q_y^2)-q_x^2}\ , \tag{16}$$

or

$$t\cdot(q_y^2-q_z^2)+q_z^2=\frac{p_x^2}{x}\ ,\quad t\cdot(q_z^2-q_x^2)=\frac{p_y^2}{y}\ ,\quad t\cdot(q_x^2-q_y^2)-q_x^2=\frac{p_z^2}{z}\ . \tag{17}$$

Multiplying equations (17) by q_x^2, q_y^2, q_z^2 respectively and summing them, we obtain

$$\frac{p_x^2q_x^2}{x}+\frac{p_y^2q_y^2}{y}+\frac{p_z^2q_z^2}{z}=0\ . \tag{18}$$

Finally, multiplying (18) by xyz follows (14).

[6] Miscellaneous Results on Tetrahedron: http://misctetrahedron.webs.com