Generalized Simson Line – Simplified Approach

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A remarkable generalization of Simson Line was done by Miguel de Guzmán[1,2]. Next we present a simplified approach.

Let ABC be an arbitrary triangle, and P an arbitrary point on its circumcircle. It is well-known that orthogonal projections Pa, Pb, Pc of P on the respective sides of triangle ABC are collinear, they define the Simson Line. De Guzmán showed that for any triple of projection directions to the respective sides of triangle ABC, there is a locus of points P, such that projection points Pa, Pb, Pc are collinear.

Let Q = (u : v : w) be an arbitrary point, see Fig. 1.

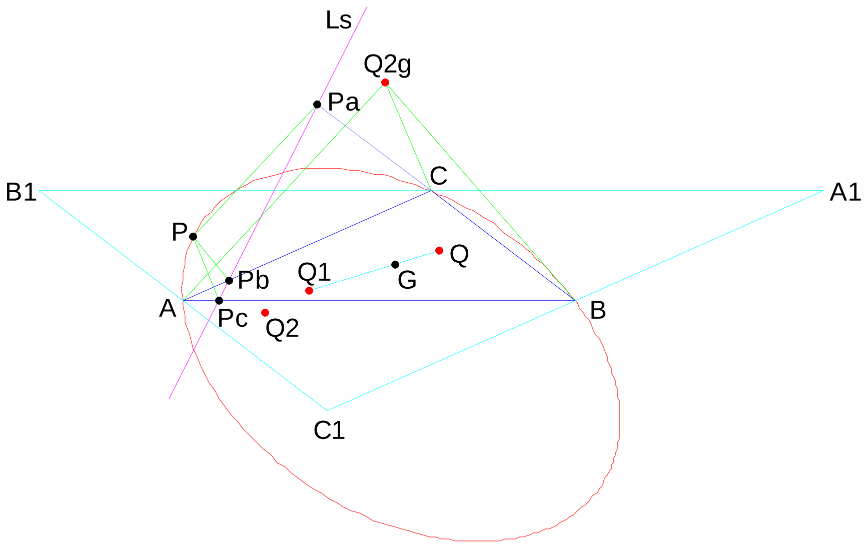

Fig. 1

The iso-Q conjugate[3] of the line at infinity is the circumconic

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[1] Miguel de Guzmán: An Extension of the Wallace-Simson Theorem: Projecting in Arbitrary Directions. Amer. Math. Monthly 106, 574-580 (1999).

[2] Pavel Pech: http://www.heldermann-verlag.de/jgg/jgg09/j9h2pech.pdf

[3] Avni Pllana: http://trisectlimacon.webs.com/Eagle_Theorem1.pdf

$$u^2yz+v^2zx+w^2xy=0\ , \tag{1}$$

the red ellipse in Fig. 1. Now we determine the triple of projection directions $\overrightarrow{da}$, $\overrightarrow{db}$, $\overrightarrow{dc}$, such that for any point P on (1), the projection points Pa, Pb, Pc are collinear.

Let A1B1C1 be the anticomplementary triangle of triangle ABC. The centroid G is common for both triangles ABC and A1B1C1. We construct point Q1 collinear with GQ, such that

$$\overline{GQ1}=2\cdot\overline{GQ}\quad . \tag{2}$$

Point Q1 in triangle A1B1C1 is equivalent with Q in triangle ABC. We construct point Q2 that is the iso-Q1 conjugate of G in triangle A1B1C1. Finally we construct point Q2g that is the iso-G, or the isotomic conjugate of Q2 in triangle ABC. Barycentric coordinates of Q2g with respect to triangle ABC, are

$$Q2g=\left(\frac{1}{v^2+w^2-u^2}:\frac{1}{w^2+u^2-v^2}:\frac{1}{u^2+v^2-w^2}\right)\ . \tag{3}$$

The sought projection directions $\overrightarrow{da}$, $\overrightarrow{db}$, $\overrightarrow{dc}$ are the cevians of point Q2g,

$$\overrightarrow{da}=AQ2g\ ,\ \ \overrightarrow{db}=BQ2g\ ,\ \ \overrightarrow{dc}=CQ2g\ . \tag{4}$$

For Q = I = (a : b : c) the incenter, circumconic (1) is the circumcircle and Q2g = H the orthocenter. For Q = G = (1 : 1 : 1) the centroid, circumconic (1) is the Steiner Ellipse and Q2g = G the centroid.

Next we present a generalization of orthopole[4]. Let ABC be an arbitrary triangle and L an arbitrary line, see Fig. 2. Let Q = (u : v : w) be an arbitrary point with corresponding circumconic (1), the red ellipse in Fig. 2. The point Q2g in Fig. 1 we denote by Hq, and Oq is the center of circumconic (1). It is simply the iso-Q conjugate of Hq, therefore

$$Oq=\left(u^2(v^2+w^2-u^2):v^2(w^2+u^2-v^2):w^2(u^2+v^2-w^2)\right)\ . \tag{5}$$

The iso-Q conjugate of the point at infinity of line L is the point Ql in Fig. 2. To the point Ql corresponds the Simson line Lsq. Let Pa, Pb, Pc be the intersection points of line L with respective lines through the vertices of triangle ABC and parallel to line Lsq. Now lines through Pa, Pb, Pc and parallel to the respective cevians of Hq are concurrent at point Pn, which lies on Lsq. Point Pn represents the generalized orthopole, or the Q-pole of line L.

In the case when circumconic (1) is an ellipse, see footnote (3), the line through Oq and parallel to Lsq intersects circumconic (1) at two points, whose corresponding tangents are parallel to line L. Such a tangent point is Pt in Fig. 2.

[4] Orthopole: http://mathworld.wolfram.com/Orthopole.html

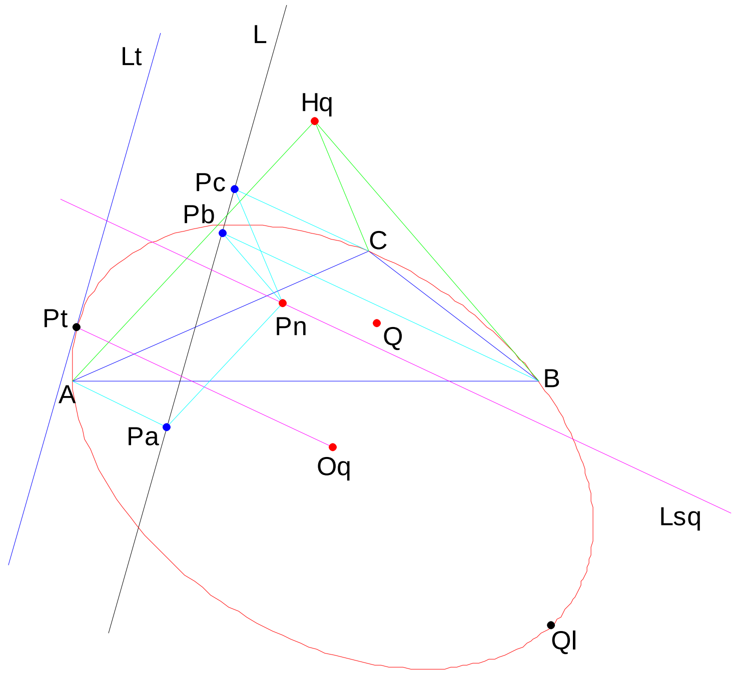

Fig. 2

Let $px + qy + rz = 0$ be the equation of the line through Oq and parallel to Lsq. Then the coordinates of an arbitrary point $P = (x : y : z)$ of that line are

$$x = t \cdot (q - r) + r \ , \ y = t \cdot (r - p) \ , \ z = t \cdot (p - q) - p \ , \tag{6}$$

where t is a real parameter. Substituting coordinates (6) in (1) and solving for t, we obtain the two intersecting points with circumconic (1). Let $Pt = (x_0 : y_0 : z_0)$ be one of those intersection points, then its corresponding tangent Lt is

$$(v^2 z_0 + w^2 y_0)x + (w^2 x_0 + u^2 z_0)y + (u^2 y_0 + v^2 x_0)z = 0 \ . \tag{7}$$