A Generalization of Ceva's Theorem for Tetrahedron

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In Fig.1 is shown a tetrahedron ABCD and an arbitrary point P inside the tetrahedron. Points Pa, Pb, Pc, Pd are the intersection points of lines AP, BP, CP, DP with the tetrahedron faces DBC, CAD, BDA, ACB respectively.

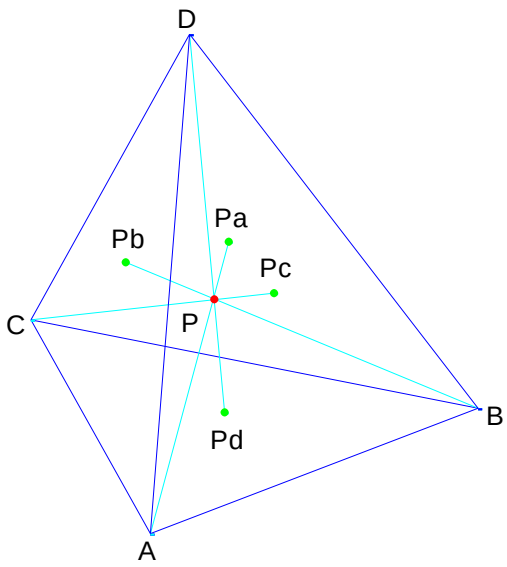

Fig.1

Because tetrahedrons CPaBA, DPaCA, BPaDA share the same altitude (i.e., from point A) we have

$$\text{Area}(CPaB) : \text{Area}(DPaC) : \text{Area}(BPaD) = \text{Vol}(CPaBA) : \text{Vol}(DPaCA) : \text{Vol}(BPaDA) \,. \quad (1)$$

The left-hand side of (1) represents the barycentric coordinates[1] of Pa, therefore we can write (1) as follows

$$Pa(D) : Pa(B) : Pa(C) = \text{Vol}(CPaBA) : \text{Vol}(DPaCA) : \text{Vol}(BPaDA) \,. \quad (2)$$

Similarly for tetrahedrons CPaBP, DPaCP, BPaDP we have

$$Pa(D) : Pa(B) : Pa(C) = \text{Vol}(CPaBP) : \text{Vol}(DPaCP) : \text{Vol}(BPaDP) \,. \quad (3)$$

[1] Barycentric Coordinates: http://mathworld.wolfram.com/BarycentricCoordinates.html

Using the property of proportions y/x = w/z = (y-w)/(x-z) , from (2) and (3) we obtain

Pa(D):Pa(B):Pa(C)=Vol(CPaBA)-Vol(CPaBP):Vol(DPaCA)-Vol(DPaCP):Vol(BPaDA)-Vol(BPaDP)

or

$$Pa(D) : Pa(B) : Pa(C) = Vol(ACBP) : Vol(CADP) : Vol(BDAP) . \quad (4)$$

Similarly for Pb, Pc, Pd we obtain

$$Pb(C) : Pb(A) : Pb(D) = Vol(BDAP) : Vol(DBCP) : Vol(ACBP) , \quad (5)$$

$$Pc(B) : Pc(D) : Pc(A) = Vol(CADP) : Vol(ACBP) : Vol(DBCP) , \quad (6)$$

$$Pd(A) : Pd(C) : Pd(B) = Vol(DBCP) : Vol(BDAP) : Vol(CADP) . \quad (7)$$

Now multiplying (4), (5), (6), and (7) we obtain

Pa(D)*Pb(C)*Pc(B)*Pd(A) : Pa(B)*Pb(A)*Pc(D)*Pd(C) : Pa(C)*Pb(D)*Pc(A)*Pd(B) = 1 : 1 : 1 . (8)

Relation (8) represents a generalization of Ceva's Theorem[2] for a tetrahedron.

Application Example:

It is well-known that a tetrahedron has an orthocenter if the feet of its altitudes coincide with the orthocenters of the respective tetrahedron faces. If the lengths of the edges a, b, c are known, we want to determine the lengths of edges d, e, f such that the tetrahedron has an orthocenter. For this purpose we lay the faces DBC, CAD, and BDA on the plane of the face ACB as shown in Fig. 2.

Since each of the quadrilaterals ABDC, BCDA, CADB has its diagonals intersecting at right angle[3], we have the following relations

$$c^2+f^2 = b^2+e^2 = a^2+d^2 = 2/3*(a^2+b^2+c^2) . \quad (9)$$

The right-most expression in (9) is chosen with the purpose to obtain a regular tetrahedron if

a = b = c . From (9) we obtain the following expressions for d, e , f

$$d = sqrt((2*b^2+2*c^2-a^2)/3) ,$$

$$e = sqrt((2*a^2+2*c^2-b^2)/3) , \quad (10)$$

$$f = sqrt((2*a^2+2*b^2-c^2)/3) .$$

[2] Ceva's Theorem : http://mathworld.wolfram.com/CevasTheorem.html
[3] Quadrilateral : http://mathworld.wolfram.com/Quadrilateral.html

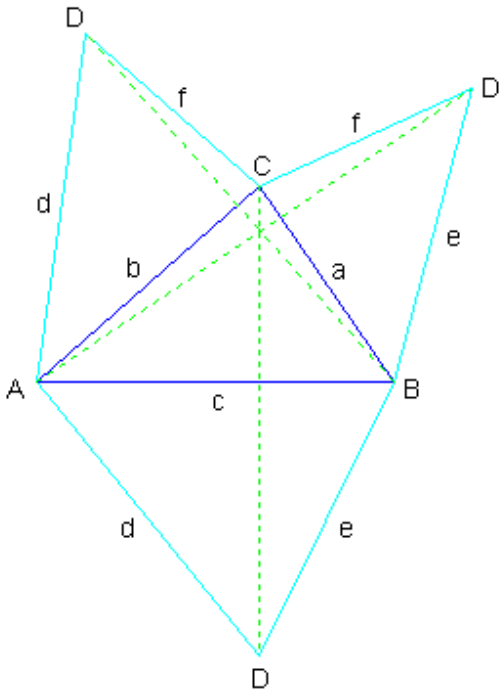

Fig. 2

Now we use the generalized Ceva's Theorem (8) to prove that a tetrahedron with edges d, e, f calculated using equations (10), has an orthocenter, or in other words its altitudes are concurrent.

The orthocenter of triangle ACB has barycentric coordinates Pd(A) : Pd(C) : Pd(B) as follows

(a^2+b^2-c^2)*(c^2+a^2-b^2) : (c^2+a^2-b^2)*(b^2+c^2-a^2) : (b^2+c^2-a^2)*(a^2+b^2-c^2) . (11)

Using (11) and (10) to calculate barycentric coordinates of orthocenters of triangles DBC, CAD, BDA , we obtain for the left-most expression of (8)

Pa(D)*Pb(C)*Pc(B)*Pd(A) =

1/9*(a^2+b^2-c^2)^2*(b^2+c^2-a^2)^2*(c^2+a^2-b^2)^2*(a^2+b^2+c^2)^2 . (12)

The same right-hand expression of (12) we obtain for Pa(B)*Pb(A)*Pc(D)*Pd(C) , and

Pa(C)*Pb(D)*Pc(A)*Pd(B), so the proof is completed.