A Generalization of the Nagel Point[1]

Avni Pllana

In Fig. 1 is shown an arbitrary triangle ABC. Let angle(CAY) = angle(BAZ) = theta , angle(ABZ) = angle(CBX) = phi , angle(BCX) = angle(ACY) = psi . Points Pa, Pb, Pc represent a certain triangle center Q = U(a,b,c) : V(a,b,c) : W(a,b,c) for the respective triangles XBC, AYC, ABZ. The homogenous lengths of the sides of triangles XBC and AYC are as follows

BC = aa = sin(phi + psi), CX = ab = sin(phi), XB = ac = sin(psi), (1)
YC = ba = sin(theta), CA = bb = sin(psi + theta), AY = bc = sin(psi). (2)

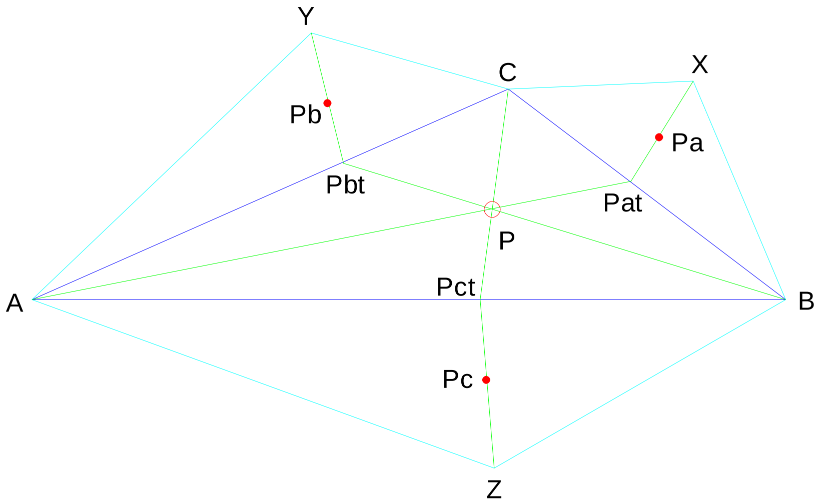


Fig. 1

Points Pat, Pbt are traces of the points Pa, Pb on the lines BC, CA respectively. The barycentric coordinates of points Pat, Pbt are

Pat = 0 : va : wa ,
Pbt = ub : 0 : wb ,

[1] Nagel Point: http://mathworld.wolfram.com/NagelPoint.html

where va = V(aa,ab,ac), wa = W(aa,ab,ac), ub = U(ba,bb,bc), wb = W(ba,bb,bc) with aa, ab, ac and ba, bb, bc defined by expressions (1), (2) .

Now lines APat, BPbt, CPct are concurrent at the point P, with barycentric coordinates

$$P = wa*ub : va*wb : wa*wb . \qquad (3)$$

The Nagel point X(8) we obtain from (3) for the special case Q = X(4), that means the orthocenter, and theta = (pi – alf)/2, phi = (pi – bet)/2, psi = (pi –gam)/2 , where alf, bet, gam are the angles at the respective vertices A, B, C of the triangle ABC.
The Gergonne point X(7), is obtained for Q = X(4), theta = -alf/2, phi = -bet/2, psi = -gam /2.
The Yff center of congruence X(174),), is obtained for Q = X(1), that means the incenter, and theta = -alf/2, phi = -bet/2, psi = -gam /2.

An interesting result is, that for Q = forward Brocard point[2] (Brf = 1/c^2 : 1/a^2 : 1/b^2), the lines APat, BPbt, CPct are concurrent at a point Pbf, only for theta = (pi – alf)/2, phi = (pi – bet)/2, psi = (pi –gam)/2. The barycentric coordinates of Pbf are

$$Pbf = 1/(cos(gam/2))^2 : 1/(cos(alf/2))^2 : 1/(cos(bet/2))^2 \quad . \qquad (4)$$

The same holds for Q = backward Brocard point (Brb = 1/b^2 : 1/c^2 : 1/a^2). The barycentric coordinates of the intersection point Pbb of the lines APat, BPbt, CPct, are

$$Pbb = 1/(cos(bet/2))^2 : 1/(cos(gam/2))^2 : 1/(cos(alf/2))^2 \quad . \qquad (5)$$

For theta = (pi – alf)/2, phi = (pi – bet)/2, psi = (pi –gam)/2, the lines APat, BPbt, CPct are concurrent even for Q = Ia = -a : b : c , keeping in mind the orientation of triangles XBC, AYC, ABZ and relations (1), (2).

In Fig. 2 is shown a generalization of the Mittenpunkt X(9). Points Ia, Ib, Ic are the excenters of the arbitrary triangle ABC. Let angle(CAY) = angle(BAZ) = theta , angle(ABZ) = angle(CBX) = phi , angle(BCX) = angle(ACY) = psi . Then lines IaX, IbY, IcZ are concurrent at a point P with barycentric coordinates

$$P = h(a,b,c,theta,phi,psi) : h(b,c,a,phi,psi,theta) : h(c,a,b,psi,theta,phi), \text{ where}$$

$$h(a,b,c,theta,phi,psi) = a*(2*s*(s-b)*(s-c)-a*b*c+area(ABC)*(b/tan(phi)+c/tan(psi)-a/tan(theta))) , (6)$$

where s = (a+b+c)/2 .

The Mittenpunkt X(9) we obtain from (6) as a limit case when theta = phi = psi = delta and delta tends to zero,

$$X(9) = a*(b+c-a) : b*(c+a-b) : c*(a+b-c) . \qquad (7)$$

[2] Brocard Points: http://mathworld.wolfram.com/BrocardPoints.html

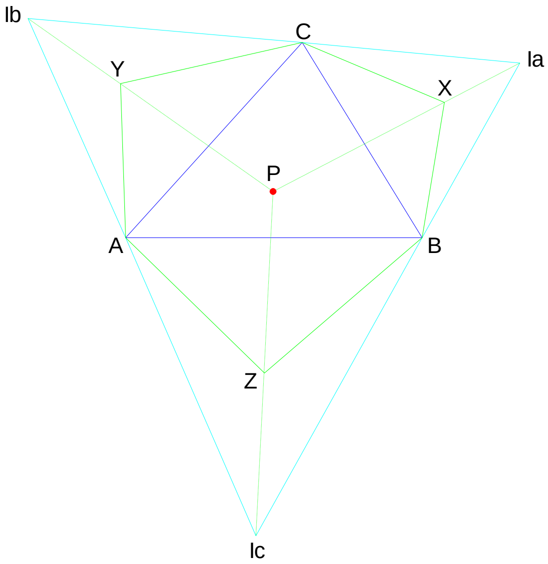
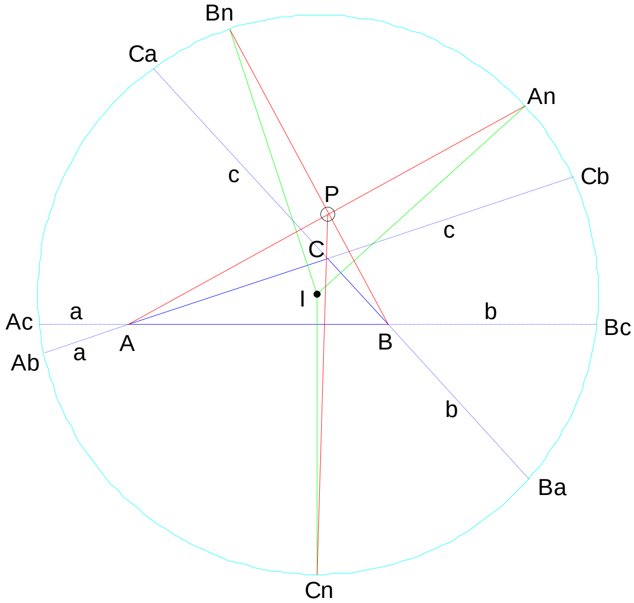


Fig. 2

In Fig. 3 is shown the Conway circle[3] for the arbitrary triangle ABC. The Conway circle is centered at the incenter I and has radius Rc = sqrt(r^2 + s^2), where r is the radius of the incircle and s is the semiperimeter of the triangle ABC.



[3] Conway Circle: http://mathworld.wolfram.com/ConwayCircle.html

Fig. 3

The rays emanating from I and perpendicular to the sides BC, CA, AB, intersect the Conway circle at points An, Bn, Cn respectively. Now lines AAn, BBn, CCn are concurrent at a point P with barycentric coordinates f(a,b,c) : f(b,c,a) : f(c,a,b) , where

$$f(a,b,c) = \tan(alf) / (Rp - r + (s-a)*\tan(alf)) , \quad (8)$$

where alf is the angle at the vertex A of the triangle ABC, and Rp = Rc. The theorem also holds for any radius Rp. For Rp = r we obtain the Gergonne point X(7), and for Rp = -r we obtain the Nagel point X(8). Triangle AnBnCn represents an extension of the Gergonne triangle.

In Fig. 4 is shown a generalization of the Isogonal Mittenpunkt[4] X(57).

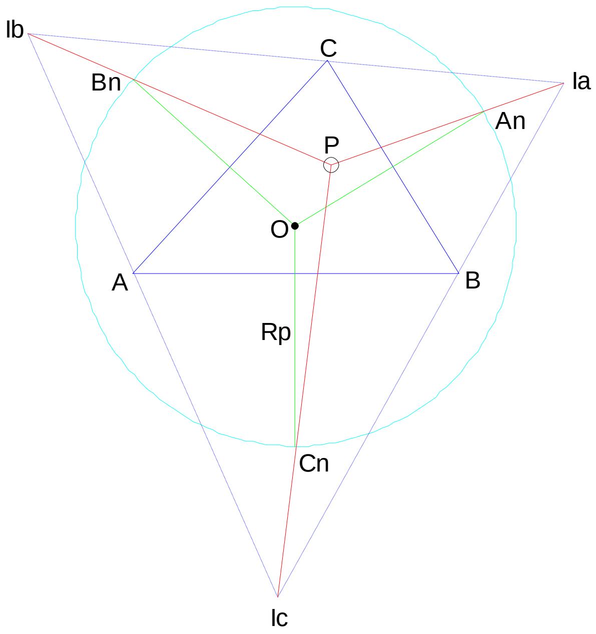


Fig. 4

The rays emanating from the circumcenter O of the arbitrary triangle ABC and perpendicular to the sides BC, CA, AB, intersect the circle with center O and arbitrary radius Rp at points An, Bn, Cn respectively. Points Ia, Ib, Ic are the excenters of the triangle ABC. Now lines IaAn, IbBn, IcCn are concurrent at a point P with barycentric coordinates f(a,b,c) : f(b,c,a) : f(c,a,b) , where

$$f(a,b,c) = a*(R*(2*a*s*(s-a)-a*b*c)+Rp*(b*s*(s-b)+c*s*(s-c)-a*s*(s-a)-a*b*c)) , \quad (9)$$

[4] Isogonal Mittenpunkt: http: //mathworld.wolfram.com/IsogonalMittenpunkt.html

where R is the circumradius and s is the semiperimeter of the triangle ABC. Expression (9) can also be written in terms the angles alf, bet, gam at the respective vertices A, B, C of the triangle ABC,

$$f(a,b,c) = a*(2*R*cos(alf)+Rp*(cos(bet)+cos(gam)-cos(alf)-1)) \ . \quad (10)$$

We obtain from (9) the Isogonal Mittenpunkt , X(57) = a/(s-a) : b/(s-b) : c/(s-c), as a special case when

$$Rp = R*(U*(s-a)-V*(s-b))/(Vp*(s-b)-Up*(s-a)) \ , \quad (11)$$

where

$$U = 2*a*s*(s-a)-a*b*c \ , \quad Up = b*s*(s-b)+c*s*(s-c)-a*s*(s-a)-a*b*c \ ,$$

$$V = 2*b*s*(s-b)-a*b*c \ , \quad Vp = c*s*(s-c)+a*s*(s-a)-b*s*(s-b)-a*b*c \ .$$

For Rp = R we obviously obtain the incenter I . The point P is on the line OI which is the Euler line of the excentral triangle IaIbIc.

The above results can be extended if we start with any point Q that lies on the line OI, instead of the circumcenter O. The barycentric coordinates of Q are f(a,b,c) : f(b,c,a) : f(c,a,b) , where

$$f(a,b,c) = s*a^2*(b^2+c^2-a^2)*t+2*S^2*a*(1-t) \ , \quad (12)$$

where s = (a+b+c)/2 , S = 2*area(ABC) , and t is a parameter that takes values in the interval (-inf, +inf) . The generalized form of (10) is as follows

$$f(a,b,c) = a*(2*Ra+Rp*(cos(bet)+cos(gam)-cos(alf)-1)) \ , \quad (13)$$

where Ra = (R*cos(alf)-r)*t+r , is the distance of point Q from the line BC. It is clear that (10) follows from (13) for t = 1 .

For t = 2 from (12) we obtain X(40), this is the circumcenter of triangle IaIbIc. Since in this case (13) yields barycentric coordinates of X(40) independently of radius Rp, we obtain the nice relation

$$cos(alf) + cos(bet) + cos(gam) = (r + R)/R \ . \quad (14)$$

In Fig. 5 is shown an arbitrary triangle ABC and an interesting triangle center P. Points Ia, Ib, Ic are the excenters of the triangle ABC, and form the excentral triangle IaIbIc. Points Ai, Bi, Ci are the incenters of triangles CBIa, ACIb, BAIc respectively. Points Ao, Bo, Co are the circumcenters of triangles CBIa, ACIb, BAIc respectively. Point O is the circumcenter of triangle ABC, and Ie is the incenter of the excentral triangle IaIbIc. Now lines AiAo, BiBo, CiCo are concurrent at the point P with barycentric coordinates u : v : w

$$u = a/(b*\cos(gam/2)-c*\cos(bet/2)) ,$$

$$v = b/(c*\cos(alf/2)-a*\cos(gam/2)) , \qquad (15)$$

$$w = c/(a*\cos(bet/2)-b*\cos(alf/2)) ,$$

where alf, bet, gam are the angles at the respective vertices A, B, C of triangle ABC.

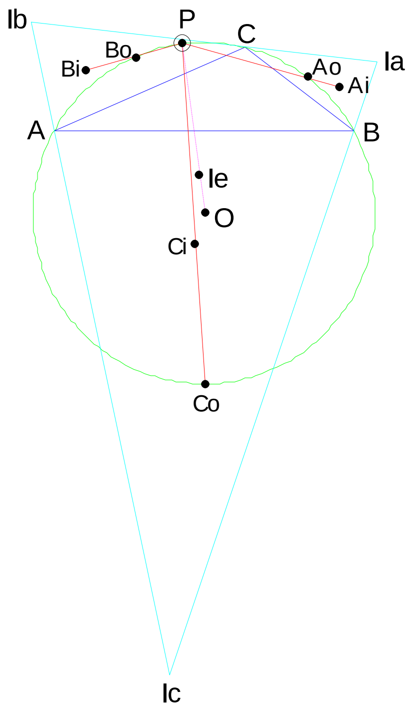


Fig. 5

It is interesting that triangle center P lies on the line OIe and on the circumcircle of the triangle ABC. The Simson line[5] that corresponds to the point P is parallel to the line that passes through the circumcenter Oe = X(40) and incenter Ie = X(164) of the excentral triangle IaIbIc. This means that reflections of the line Lhe in the sidelines of triangle ABC intersect at the center P, where Lhe is the line through orthocenter H = X(4) of triangle ABC and parallel to the line OeIe.

The above theorem can be generalized if we take instead of the incenter, that means instead Ai, Bi, Ci, any other triangle center Q = U(a,b,c) : V(a,b,c) : W(a,b,c). Then lines AqAo, BqBo, CqCo are concurrent at the point Pq with barycentric coordinates u : v : w

[5] Simson line: http://mathworld.wolfram.com/SimsonLine.html

u = a/(b*Xc-c*Xb) , v = b/(c*Xa-a*Xc) , w = c/(a*Xb-b*Xa) , (16)

where

Xa = U(aa,bb,cc) , Xb = V(aa,bb,cc) , Xc = W(aa,bb,cc) , where

aa = cos(alf/2) , bb = cos(bet/2) , cc = cos(gam/2) .

For Q = X(8) , that means the Nagel point X(8) = b+c-a : c+a-b : a+b-c , we have

Xa = bb+cc-aa , Xb = cc+aa-bb , Xc = aa+bb-cc .

The point Pq lies on the circumcircle of triangle ABC.
The theorem holds even for Q = Ia = -a : b : c , keeping in mind the orientation of triangles XBC, AYC, ABZ from Fig. 1 .

In Fig. 6 is shown a generalization of the de Longchamps point[6] X(20). Points Ia, Ib, Ic are the excenters of the arbitrary triangle ABC. Let Ia(t*ra) be the circle with center at point Ia and radius t*ra, where ra is the exradius and t is a real parameter. Point Ab is the intersection of the circle Ia(t*ra) with the ray emanating from Ia and perpendicular to the line AB. Point Ac is the intersection of the circle Ia(t*ra) with the ray emanating from Ia and perpendicular to the line AC. Lines BAc and CAb intersect at a point A1. Similarly we construct points B1 and C1.

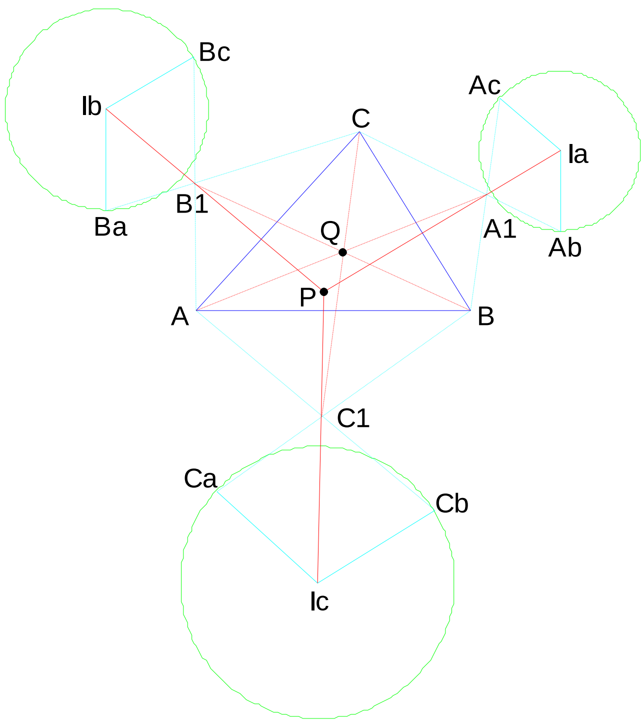


Fig. 6

[6] De Longchamps point: http://mathworld.wolfram.com/deLongchampsPoint.html

Now lines IaA1, IbB1, IcC1 are concurrent at a point P with barycentric coordinates u : v : w , where

$$u = a*(\cos(alf)-t*\cos(bet)*\cos(gam)) , \qquad (17)$$

and alf, bet, gam are the angles at the respective vertices A, B, C of triangle ABC.

Furthermore lines AA1, BB1, CC1 also are concurrent at a point Q with barycentric coordinates u : v : w , where

$$u = a/(1-t*\cos(alf)) . \qquad (18)$$

For t = 1, t = ½, t = -1 from (17) we obtain triangle centers X(20), X(376), X(2) respectively. It is clear that from (17) we obtain points that lie on the Euler line of triangle ABC.

For t = 1, t = ½, t = -1 from (18) we obtain triangle centers X(8), X(1000), X(7) respectively.

Since for t = -1 we obtain from (17) the centroid G = 1 : 1 : 1 of triangle ABC, we derive the following nice relations

$$\begin{aligned} area(ABC)/R &= a*(\cos(alf) + \cos(bet)*\cos(gam)) , \\ &= b*(\cos(bet) + \cos(gam)*\cos(alf)) , \qquad (19) \\ &= c*(\cos(gam) + \cos(alf)*\cos(bet)) , \end{aligned}$$

where R is the circumradius of triangle ABC.