Eagle Theorem

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Let ABC be an arbitrary triangle, and A1B1C1 its anticomplementary triangle[1] as shown in Fig. 1.

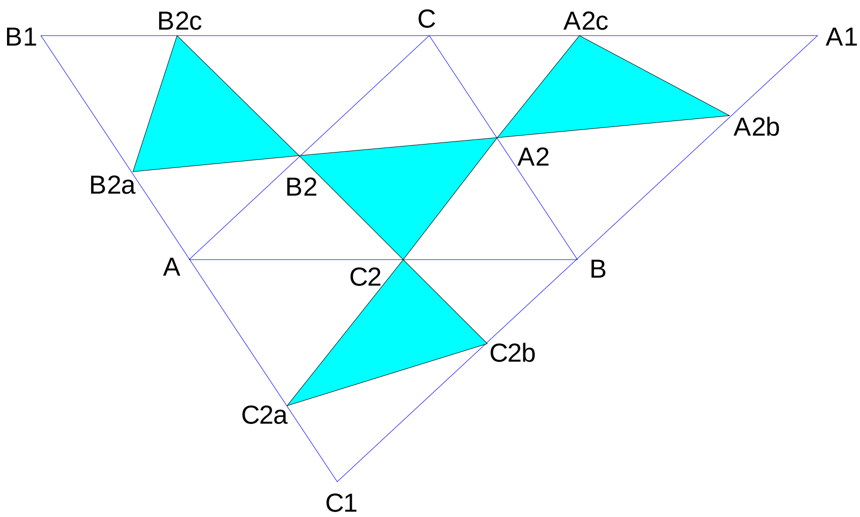

Fig. 1

Let A2, B2, C2 be arbitrary points on lines BC, CA, AB respectively. Let B2c, C2b be the intersection points of line B2C2 with lines A1B1, C1A1. Similarly we obtain points C2a, A2c and A2b, B2a. Then triangles A2B2C2, A2A2bA2c, B2B2cB2a, C2C2aC2b have equal area

$$S_{A_2B_2C_2} = S_{A_2A_{2b}A_{2c}} = S_{B_2B_{2c}B_{2a}} = S_{C_2C_{2a}C_{2b}} \ . \tag{1}$$

[1] Anticomplementary Triangle: http://mathworld.wolfram.com/AnticomplementaryTriangle.html

If points A2, B2, C2 are traces of cevians of some arbitrary point P on the respective sides of triangle ABC, then

$$\overline{AB_{2a}} = \overline{AC_{2a}} \ , \ \overline{BC_{2b}} = \overline{BA_{2b}} \ , \ \overline{CA_{2c}} = \overline{CB_{2c}} \ . \qquad (2)$$

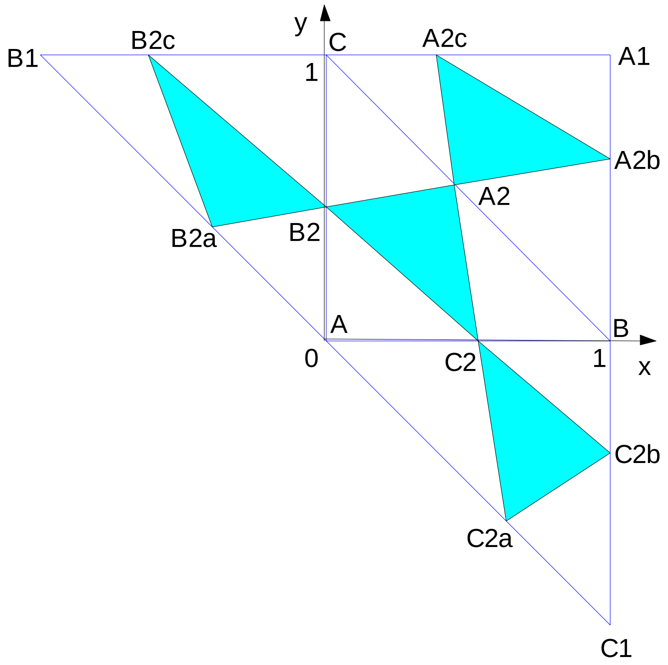

Fig. 2

For the sake of simplicity, without loss of generality Fig. 1 can be transformed to Fig. 2 using affine transformations.

Let

$$A_2 = [x_a \ , 1 - x_a], \ B_2 = [0 \ , y], \ C_2 = [x \ , 0] \ . \qquad (3)$$

From (3) we have

$$2 \cdot S_{A_2B_2C_2} = \begin{vmatrix} x_a & 1 - x_a & 1 \\ 0 & y & 1 \\ x & 0 & 1 \end{vmatrix} = xy - yx_a + x_a x - x \ . \qquad (4)$$

From (3) and Fig. 2 we also have

$$2 \cdot S_{A_2A_{2b}A_{2c}} = \begin{vmatrix} x_a & 1-x_a & 1 \\ 1 & (1-x_a).\dfrac{1-y}{x_a} & 1 \\ x_a \cdot \dfrac{1-x}{1-x_a} & 1 & 1 \end{vmatrix} = xy - yx_a + x_a x - x \ . \qquad (5)$$

From (4) and (5) follows (1).

In the case triangle $A_2B_2C_2$ is a cevian triangle of some arbitrary point P, then according to (3), Fig. 2 and Ceva's Theorem we have

$$x \cdot (1-x_a) \cdot (1-y) = (1-x) \cdot x_a \cdot y \ . \qquad (6)$$

From (3) and Fig. 2 we have

$$\overline{BC_{2b}} = \frac{y \cdot (1-x)}{x} \ , \quad \overline{BA_{2b}} = \frac{(1-x_a) \cdot (1-y)}{x_a} \ . \qquad (7)$$

From (6) and (7) follows (2).

Next we present some results on circumconics[2] . It is well-known that the isogonal conjugate of the line at infinity is the circumcircle of arbitrary triangle ABC, with the following equation in barycentric coordinates

$$a^2yz + b^2zx + c^2xy = 0 \ . \qquad (8)$$

The isotomic conjugate of the line at infinity is the Steiner Circumellipse[3], with the following equation

$$yz + zx + xy = 0 \ . \qquad (9)$$

From equations (8) and (9) it is clear that the coefficients of the equation are respectively the squared coordinates of incenter I = (a:b:c) and centroid G = (1:1:1) of triangle ABC. According to the generalization of the isotomic and isogonal conjugate mapping[4,5] for an arbitrary point P = (d:e:f) we can write

$$d^2yz + e^2zx + f^2xy = 0 \ . \qquad (10)$$

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[2] Circumconics: http://mathworld.wolfram.com/Circumconic.html

[3] Steiner Circumellipse: http://mathworld.wolfram.com/SteinerCircumellipse.html

[4] Miscellaneous Results on Tetrahedron: http://misctetrahedron.webs.com/

[5] Incenter and Centroid: http://trisectlimacon.webs.com/Incenter_Centroid1.pdf

Using the equation x+y+z = 1 for the absolute barycentric coordinates, we have

$$z = 1 - x - y \ . \tag{11}$$

From (10) and (11) we obtain

$$e^2x^2 + (d^2 + e^2 - f^2)xy + d^2y^2 - e^2x - d^2y = 0 \ . \tag{12}$$

The discriminant of equation (12) is

$$D = (d^2 + e^2 - f^2)^2 - 4d^2e^2 \ . \tag{13}$$

For D < 0 the line at infinity is mapped to an ellipse, for D = 0 we obtain a parabola, and for D > 0 we obtain a hyperbola. These results are illustrated in Fig. 3.

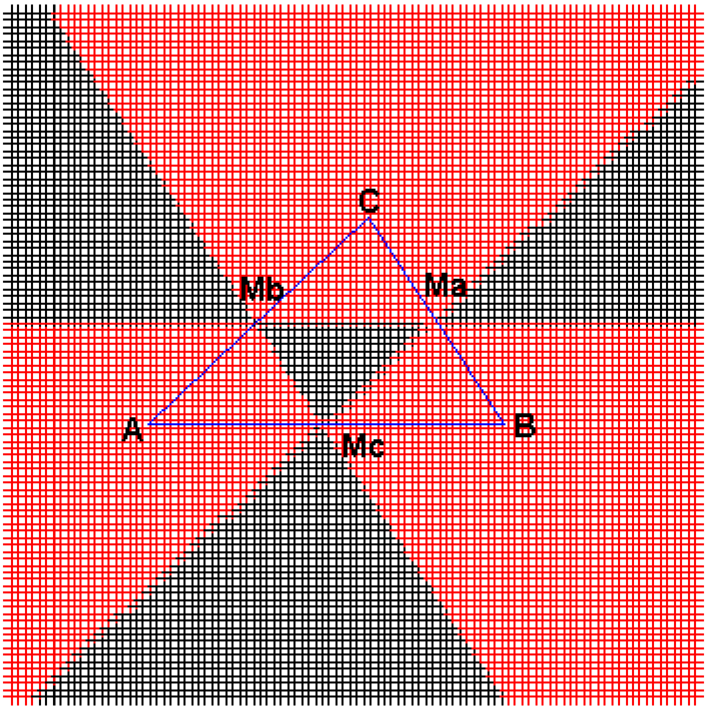

Fig. 3

The triangle MaMbMc is the medial triangle of triangle ABC. For a point P = (d:e:f) in the black region the line at infinity is mapped to an ellipse, for points in the red region we obtain a hyperbola. The incenter of triangle ABC lies always inside triangle MaMbMc, in fact it is the Nagel Point of triangle MaMbMc.