Deltoid as locus of orthopoles

Avni Pllana

Let ABC be an arbitrary triangle, and let P be an arbitrary point on the circumcircle of triangle ABC, see Fig. 1.

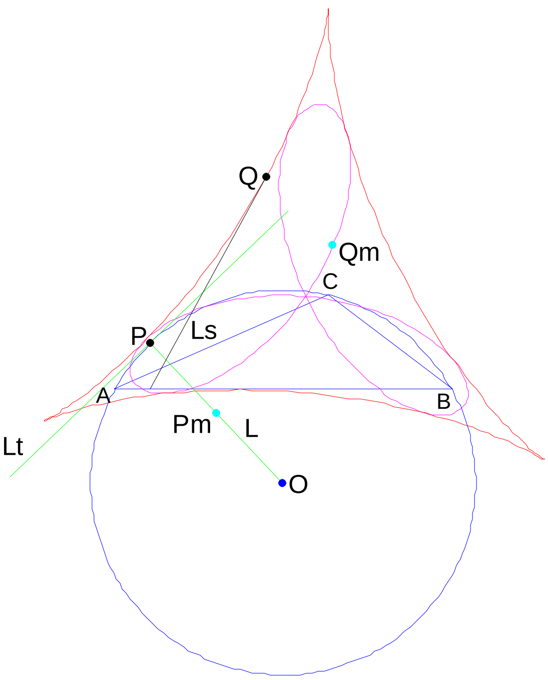

Fig. 1

Let Lt be the tangent line through P. The orthopole[1] of Lt with respect to triangle ABC is the point Q. As point P moves along the circumcircle, the point Q forms a deltoid[2], the red curve in Fig. 1. The Simson line[3] Ls that corresponds to P, tangents the deltoid exactly at point Q.

If we repeat the construction with points Pm, that means the middle points of radius OP, the corresponding orthopoles Qm form a trifolium[4], the magenta curve in Fig. 1.

Points Pm, Qm, Q, P form a parallelogram, so we have $\overline{PPm} = \overline{QQm} = R/2$, where R is the circumradius of triangle ABC.

[1] Orthopole: http://mathworld.wolfram.com/Orthopole.html
[2] Deltoid: http://mathworld.wolfram.com/Deltoid.html
[3] Simson Line: http://mathworld.wolfram.com/SimsonLine.html
[4] Trifolium: http://mathworld.wolfram.com/Trifolium.html

Instead of tangent lines on points P, Pm we can use lines that pass through the circumcenter O, but orthogonal to OP. In that case the corresponding orthopoles Qo generate the nine-point circle[5] of triangle ABC, see Fig. 2, the magenta circle.

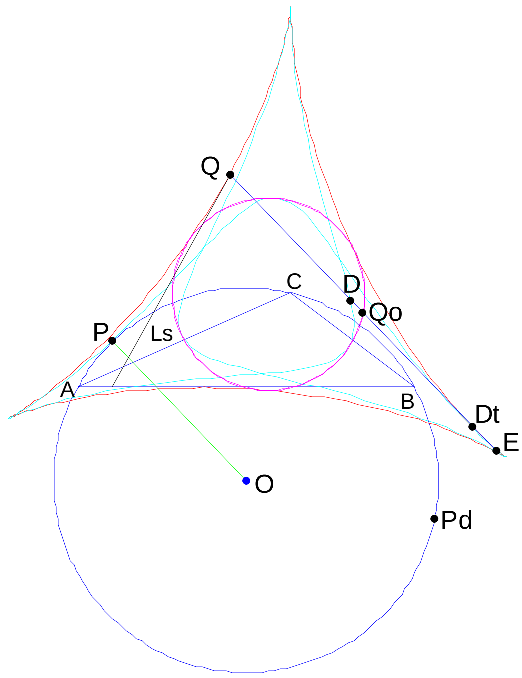

Fig. 2

Again, points O, Qo, Q, P form a parallelogram, so we have $\overline{PO} = \overline{QQo} = R$. The line QQo tangents the deltoid at point Dt, and intersects it again at point E. It is well-known that $\overline{QE} = 2R$. In order to determine point Dt, we first construct the point Pd on the circumcircle, whose Simson Line is the line QE, then Dt is the orthopole of the tangent through Pd. Next we construct point D, which is the middle point of segment QDt. As P moves along the circumcircle, point D generates a bell deltoid[6], the cyan curve in Fig. 2.

Next we construct points An, Bn, Cn where triangle sides BC, CA, AB respectively tangent the deltoid, see Fig. 3.

[5] Nine-point circle: http://mathworld.wolfram.com/Nine-PointCircle.html
[6] Bell deltoid: http://trisectlimacon.webs.com/side_bisector_triangle1.pdf

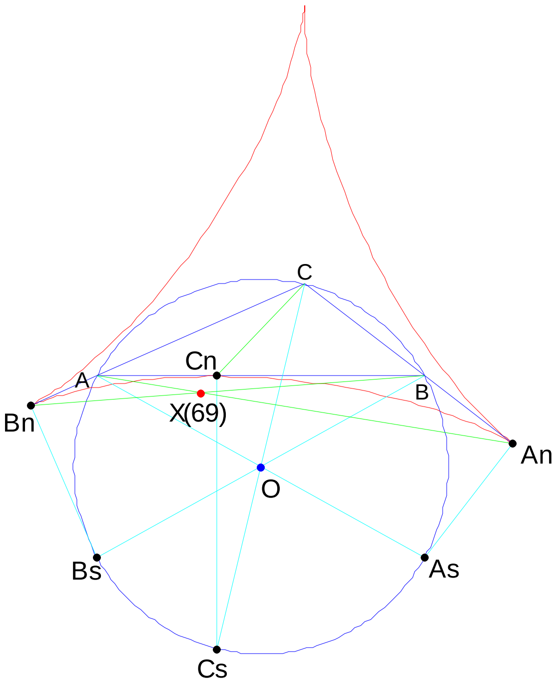

Fig. 3

First we construct points As, Bs, Cs whose Simson Lines are the respective sides of triangle ABC. In fact they are simply the opposite points of triangle vertices A, B, C with respect to circumcenter O. Now points An, Bn, Cn are orthopoles of tangents through As, Bs, Cs respectively, or simply the orthogonal projection of As, Bs, Cs on the respective sides BC, CA, AB. Triangles ABC and AnBnCn are perspective with the perspectivity center X(69), which is the symmedian point of the anticomplementary triangle.

Let P be an arbitrary point on the circumcircle, and Lt its corresponding tangent. Let An, Bn, Cn be the intersection points of lines through P and orthogonal to respective sides of triangle ABC with the circumcircle, see Fig. 4. Triangle AnBnCn is congruent with triangle ABC. Let Lm be the axis of symmetry between triangles AnBnCn and ABC. Lines Lt and Lm intersect at a point Q. As P moves along the circumcircle, Q generates the red curve in Fig. 4. It is exactly the epi spiral of trifolium, that means it is the inverse of trifolium with respect to the circumcircle.

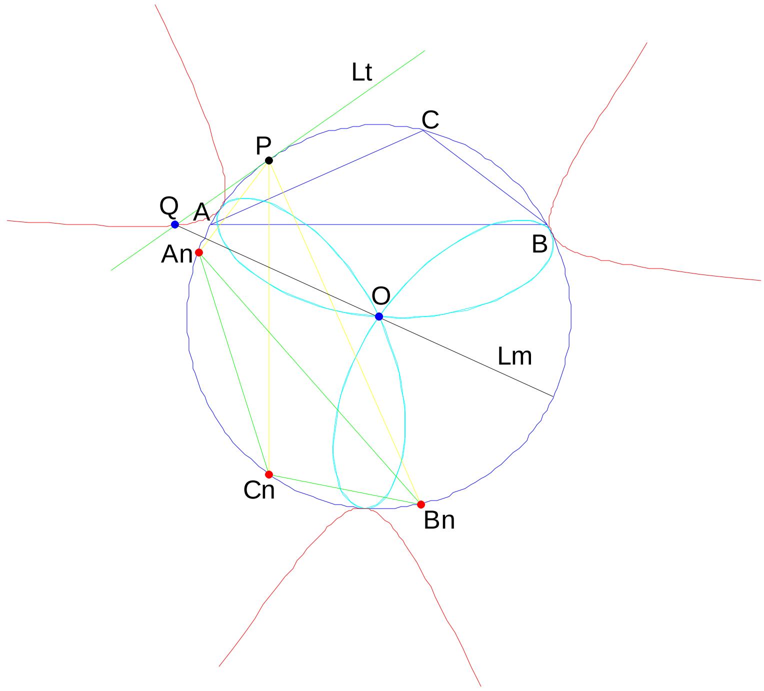

Fig. 4