Circle Connections

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Let ABC be an arbitrary triangle, and IaIbIb, AmBmCm its respectively excentral, median triangles as shown in Fig. 1.

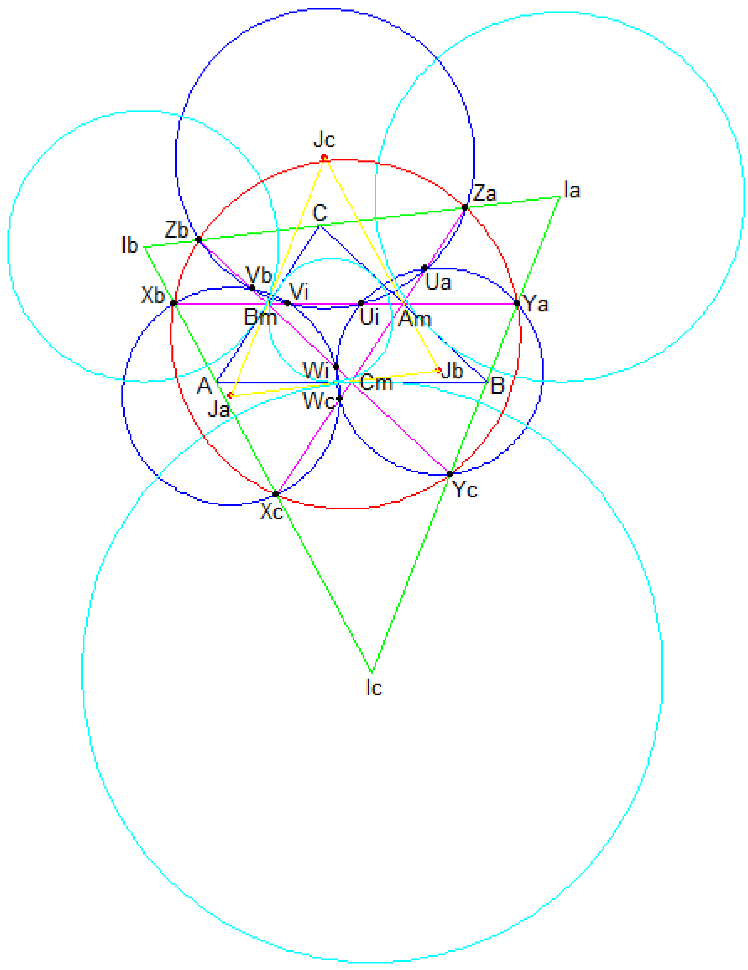

Fig. 1

Line BmCm intersects lines IaIb, IcIa at points Zb, Yc respectively. Line CmAm intersects lines IbIc, IaIb at points Xc, Za respectively. Line AmBm intersects lines IcIa, IbIc at points Ya, Xb respectively. Through points Xb, Xc, Yc, Ya, Za, Zb passes the circle Circ(Sp,Rs), where Sp is the Spieker Center X(10) of triangle ABC, or the incenter of median triangle AmBmCm, and

$$R_s = \frac{1}{2 \cdot s}\sqrt{s^4 + S^2} \ , \qquad (1)$$

where s and S are the semi-perimeter and area of triangle ABC.

Points Zb, Vb are orthogonal projections of point A on lines IaIb, BIb respectively. Points Vi, Ui are orthogonal projections of point C on lines BIb, AIa respectively. Points Ua, Za are orthogonal projections of point B on lines AIa, IaIb respectively. Through points Zb, Vb, Vi, Ui, Ua, Za passes the circle Circ(Jc,Rc), where

$$J_c = (b-c\ :\ a-c\ :\ a+b) \text{ and } R_c = \frac{1}{2}\sqrt{\frac{a\cdot c\cdot(b+c-a)}{a+b-c} + b\cdot(a-c)}\ . \tag{2}$$

Furthermore,

$$ZbVb = ViUi = UaZa = \frac{1}{2}(a+b-c)\ . \tag{3}$$

Similar relations hold for points Xc, Wc, Wi, Vi, Vb, Xb and Ya, Ua, Ui, Wi, Wc, Yc, so we have Circ(Ja,Ra) and Circ(Jb,Rb), where

$$J_a = (b+c : c-a : b-a) \text{ and } R_a = \frac{1}{2}\sqrt{\frac{b\cdot a\cdot(c+a-b)}{b+c-a} + c\cdot(b-a)}\ , \tag{4}$$

$$XcWc = WiVb = ViXb = \frac{1}{2}(b+c-a)\ , \tag{5}$$

$$J_b = (c-b : c+a : a-b) \text{ and } R_b = \frac{1}{2}\sqrt{\frac{c\cdot b\cdot(a+b-c)}{c+a-b} + a\cdot(c-b)}\ , \tag{6}$$

$$YaUi = UaWc = WiYc = \frac{1}{2}(c+a-b)\ . \tag{7}$$

Triangle JaJbJc is similar to the excentral triangle IaIbIc, where the center of similitude is the centroid G of triangle ABC.

We have further relations from Fig. 1,

$$AmYa = AmZa = \frac{a}{2}\ ,\ BmZb = BmXb = \frac{b}{2}\ ,\ CmXc = CmYc = \frac{c}{2}\ . \tag{8}$$

From the present results we conclude that Circ(Sp,Rs) is the Taylor Circle[1] of excentral triangle IaIbIc, and the Conway Circle[2] of median triangle AmBmCm.

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[1] Taylor Circle: http://mathworld.wolfram.com/TaylorCircle.html

[2] Conway Circle: http://mathworld.wolfram.com/ConwayCircle.html

Let us now discuss the intersection of two arbitrary circles with radius r_1 respectively r_2, and with the centers distance d. If $r_1 + r_2 > d$, then there are two intersection points. In the case $r_1 + r_2 < d$, the circles do not intersect in the common sense, but we still quite naturally obtain two points that have an interesting geometric meaning, see Fig. 2

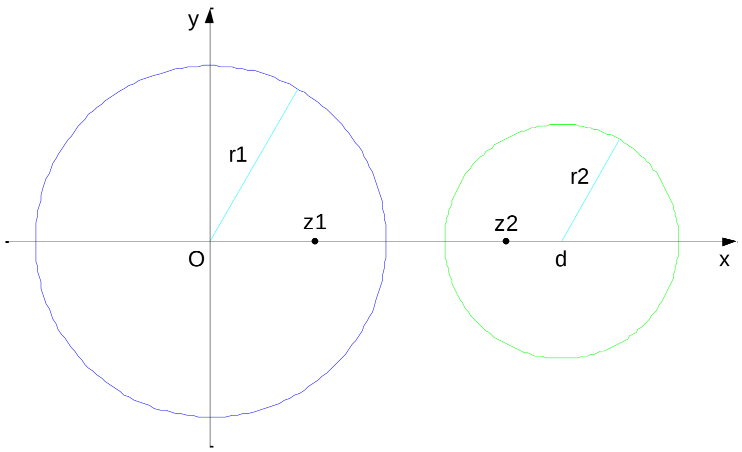

Fig. 2

In order to determine the intersection points of the two circles shown in Fig. 2, we write the following system of equations

$$x^2 + y^2 = r_1^2 \quad , \tag{9}$$

$$(x - d)^2 + y^2 = r_2^2 \quad . \tag{10}$$

From (9) and (10) we obtain points $z_1 = x + i \cdot y$, $z_2 = x - i \cdot y$, where

$$x = \frac{r_1^2 + d^2 - r_2^2}{2 \cdot d} \quad , \tag{11}$$

$$y = \frac{2}{d}\sqrt{s \cdot (s - r_1) \cdot (s - r_2) \cdot (s - d)} \quad , \quad s = \frac{r_1 + r_2 + d}{2} \quad . \tag{12}$$

From (12) follows that for $r_1 + r_2 < d$, then y is an imaginary quantity and therefore points z_1, z_2 lie on the x-axis, and we call them generalized intersection points of two circles.

The radical line[3] of the two circles passes through the point $z = \frac{z_1 + z_2}{2}$. Every circle with center on the radical line and that passes through z_1, z_2 in the case $r_1 + r_2 < d$, is orthogonal to the two circles.

Now we can say that Circ(Sp,Rs) in Fig. 1 (the red circle), passes through the generalized intersection points of excircles of triangle ABC. Points Ja, Jb, Jc lie on the respective radical lines of excircles of triangle ABC. Lines JbJc, JcJa, JaJb are radical lines of the incircle and respective excircles of triangle ABC. Point pairs {Ui, Ua}, {Vi, Vb}, {Wi, Wc} are generalized intersection points of the incircle and respective excircles of triangle ABC.

Last but not least, let we have a look at the arbitrary triangle ABC and its side-bisector reflected triangle A', B', C' shown in Fig. 3.

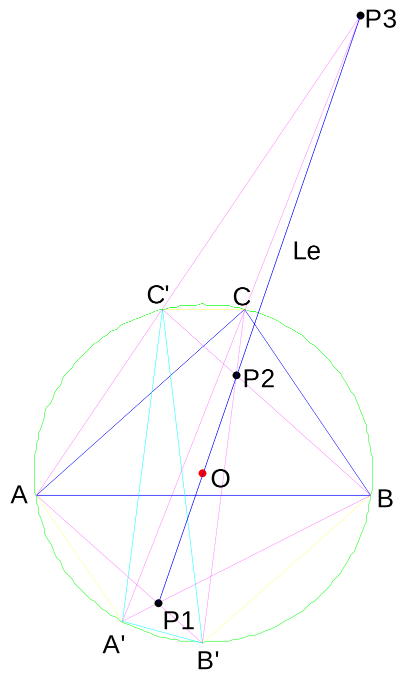

Fig. 3

[3] Radical line: http://mathworld.wolfram.com/RadicalLine.html

Vertices A', B', C' have barycentric coordinates as follows

$$\begin{aligned} A' &= (\ a^2 : c^2 - b^2 : b^2 - c^2) \\ B' &= (c^2 - a^2 : \ b^2 : a^2 - c^2) \\ C' &= (b^2 - a^2 : a^2 - b^2 : \ c^2) \end{aligned} \quad . \qquad (13)$$

According to the Pascal's Theorem[4] let P1, P2, P3 be the intersection points of the line pairs {AB', A'B}, {BC', B'C}, {CA', C'A} respectively. Points P1, P2, P3 are collinear and lie exactly on the Euler line[5] Le of triangle ABC.

[4] Pascal's Theorem: http://www.mathpages.com/home/kmath543/kmath543.htm

[5] Euler line: http://mathworld.wolfram.com/EulerLine.html