An Interesting Triangle Center

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In Fig.1 is shown an arbitrary triangle ABC and an arbitrary point P. Let angle bisectors of <BPC, <CPA, <APB intersect sides BC, CA, AB at points D, E, F respectively. Then lines AD, BE, CF are concurrent at a point Qp.

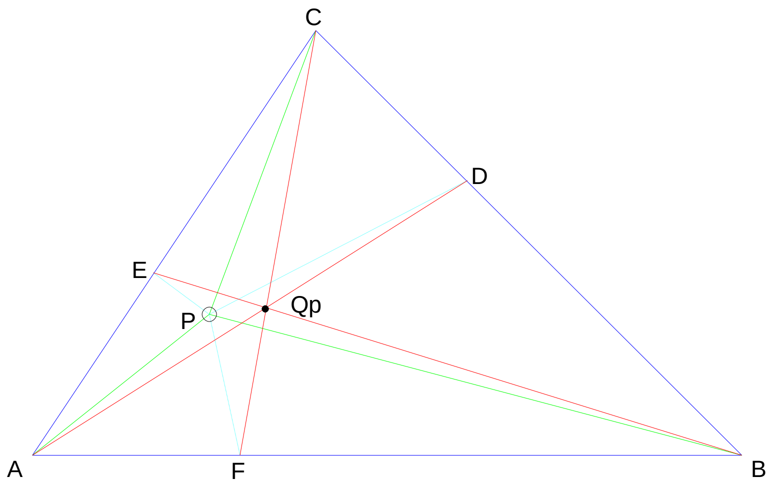

Fig. 1

According to Ceva's theorem lines AD, BE, CF are concurrent if the following condition is satisfied

$$(BD / DC) * (CE / EA) * (AF / FB) = 1 . \qquad (1)$$

Using the angle bisector theorem we have the following relations

$$BD / DC = PB / PC , \quad CE / EA = PC / PA , \quad AF / FB = PA / PB . \qquad (2)$$

Substitution of relations (2) in the left-hand side of condition (1), equals the right-hand side of (1). In other words condition (1) is satisfied simply by the way of construction of points D, E, F.

Next we will derive barycentric coordinates (U:V:W) of center Qp as function of barycentric coordinates (u : v : w) of point P[1].

From relations (2) follows

$$U : V : W = 1/PA : 1/PB : 1/PC \quad . \qquad (3)$$

Let da, db, dc be the distances of point P from sides BC = a, CA = b, AB = c respectively. Then we have

PA = 1/sin(alf)*sqrt(db^2+dc^2+2*db*dc*cos(alf)) ,

PB = 1/sin(bet)*sqrt(dc^2+da^2+2*dc*da*cos(bet)) , (4)

PC = 1/sin(gam)*sqrt(da^2+db^2+2*da*db*cos(gam)),

where alf, bet, gam are the angles of triangle ABC at vertices A, B, C respectively. According to the definition of barycentric coordinates, we have the following relations for the coordinates (u : v : w) of point P

da = 2*u/a , db = 2*v/b , dc = 2*w/c . (5)

Substituting (5) in (4) we obtain

PA = 1/area(ABC) * sqrt(c^2*v^2+b^2*w^2+v*w*(c^2+b^2-a^2)) ,

PB = 1/area(ABC) * sqrt(a^2*w^2+c^2*u^2+w*u*(a^2+c^2-b^2)) , (6)

PC = 1/area(ABC) * sqrt(b^2*u^2+a^2*v^2+u*v*(b^2+a^2-c^2)) .

Let X = PA * area(ABC) , Y = PB * area(ABC) , Z = PC * area(ABC) . Then from (3) and (6) follows

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[1] Barycentric Coordinates: http://mathworld.wolfram.com/BarycentricCoordinates.html

$$U : V : W = 1/X : 1/Y : 1/Z \quad . \qquad (7)$$

For the centroid G = (1 : 1 : 1) the directed distance kx between Qg and sideline BC of the reference triangle (a,b,c) = (6,9,13) according to the Encyclopedia of Triangle Centers[2] is

$$kx = 1.623279272390 \quad . \qquad (8)$$

The fixed point of the transformation defined by equations (6) and (7) is the first isogonic center, or Fermat point X(13).

Let U = p : q : r be an arbitrary point, and Ua, Ub, Uc its traces on the sides BC, CA, AB respectively. The reflection of cevians PUa, PUb, PUc in the angle bisectors PD, PE, PF intersect the sides BC, CA, AB at points Ua1, Ub1, Uc1 respectively. The lines AUa1, BUb1, CUc1 are concurrent at a point Up with barycentric coordinates h(a,b,c,u,v,w,p,q,r) : h(b,c,a,v,w,u,q,r,p) : h(c,a,b,w,u,v,r,p,q), where

$$h(a,b,c,u,v,w,p,q,r) = 1/(p*X^2) \text{ , or}$$

$$h(a,b,c,u,v,w,p,q,r) = 1/(p*(c^2*v^2+b^2*w^2+v*w*(c^2+b^2-a^2))) \quad . \qquad (9)$$

Expression (9) represents the **P-isogonal conjugate** of U.

In Fig. 2 is shown another triangle center P of an arbitrary triangle ABC. We extend side AB to the right and to the left by a length d, and obtain points Ab, Ba respectively. This means B is between A and Ab, whereas A is between B and Ba. We also extend sides BC, CA to the right and to the left by the same length d, and obtain points Bc, Cb, and Ca, Ac respectively.

[2] Encyclopedia of Triangle Centers : http://faculty.evansville.edu/ck6/encyclopedia/

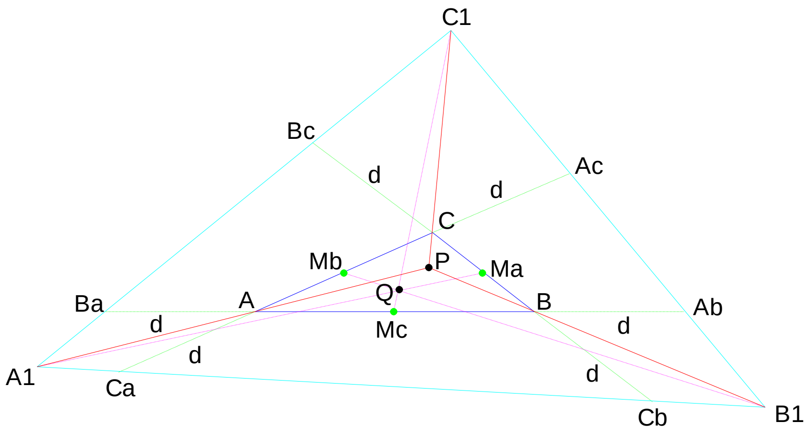

Fig. 2

Lines AbAc, BcBa intersect at point C1, lines AbAc, CaCb intersect at point B1, and lines BcBa, CaCb intersect at point A1. Now lines A1A, B1B, C1C are concurrent at a point P.

Barycentric coordinates of points Ab, Ac, Bc, Ba, Ca, Cb are

$$\begin{aligned} Ab &= -d : c+d : 0 \ , \\ Ac &= -d : 0 : b+d \ , \\ Bc &= 0 : -d : a+d \ , \\ Ba &= c+d : -d : 0 \ , \\ Ca &= b+d : 0 : -d \ , \\ Cb &= 0 : a+d : -d \ . \end{aligned} \qquad (10)$$

From (10) follow barycentric coordinates of center P

$$P = 1/(d*(b+c-a)+b*c) : 1/(d*(c+a-b)+c*a) : 1/(d*(a+b-c)+a*b) \ . \qquad (11)$$

For d = (a+b+c)/2 the directed distance kx between P and sideline BC of the reference triangle (a,b,c) = (6,9,13) according to the Encyclopedia of Triangle Centers is

$$kx = 1.173344008816 \ . \qquad (12)$$

Let Ma, Mb, Mc be the midpoints of sides BC, CA, AB of triangle ABC shown in Fig. 2. Then lines A1Ma, B1Mb, C1Mc are concurrent at a point Q with barycentric coordinates f(a,b,c) : f(b,c,a) : f(c,a,b), where

$$f(a,b,c) = a*(b+d)*(c+d)*((b+c-a)*d^2+b*c*(a+2*d)) . \qquad (13)$$

In Fig. 3 is shown another triangle center P of an arbitrary triangle ABC. We extend side AB to the right and to the left by a length d and e, to obtain points Ab, Ab1 and Ba, Ba1 respectively. This means B is between A and Ab, whereas A is between B and Ba, further Ab is between B and Ab1, and Ba is between A and Ba1. The segment BAb has the length d , and segment AbAb1 has the length e . We also extend sides BC, CA to the right and to the left by the same length d and e, and obtain points Bc, Bc1, Cb, Cb1, and Ca, Ca1, Ac, Ac1 respectively. Further A1 is the intersection point of lines BAc, CAb, and A2 is the intersection point of lines AbAc1, AcAb1. In the same way we obtain points B1 and B2 as intersection of lines ABc, CBa, and BaBc1, BcBa1 respectively. Points C1 and C2 are intersection points of lines ACb, BCa, and CaCb1, CbCa1 respectively. Now lines AA2, BB2, CC2 are concurrent at a point P.

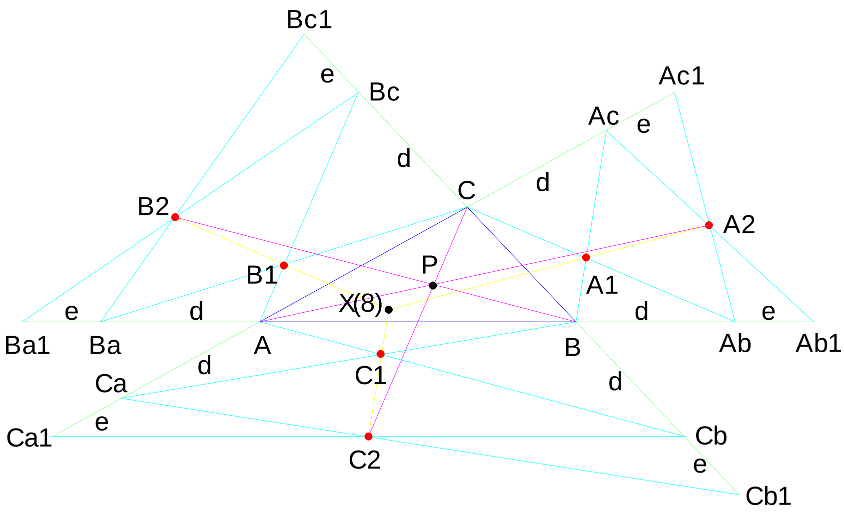

Fig. 3

The center P has the following barycentric coordinates

$$P = a/((a+d)*(a+d+e)) : b/((b+d)*(b+d+e)) : c/((c+d)*(c+d+e)) \ . \qquad (14)$$

An interesting result is that the lines A2A1, B2B1, C2C1 are parallel to the cevians of the incenter I = a : b : c and pass through the traces of the isotomic conjugate of the incenter. Lines A2A1, B2B1, C2C1 are concurrent at the Nagel point X(8), with barycentric coordinates

$$Na = b+c-a : c+a-b : a+b-c \quad , \qquad (15)$$

independently of the lengths d and e.

Let U = x : y : z be an arbitrary point, and Ua, Ub, Uc its traces on the sides AbAc, BcBa, CaCb respectively. Let the points Ua1, Ub1, Uc1 be symmetric to the points Ua, Ub, Uc with respect to the midpoints of segments AbAc, BcBa, CaCb respectively. The lines AUa1, BUb1, CUc1 are concurrent at a point Ud with barycentric coordinates

$$Ud = (a/(a+d))^2/x : (b/(b+d))^2/y : (c/(c+d))^2/z \quad . \qquad (16)$$

Expression (16) represents the **d-isotomic conjugate** of U.

In Fig. 4 is shown the so-called **parallelepiped mapping**. Let P = x : y : z be an arbitraty point, and Pa, Pb, Pc its traces on the sides BC, CA, AB respectively . Let Pa1 be the intersection point of lines passing through Pb, Pc and parallel to the cevians CPc, BPb respectively. The same way we construct points Pb1 and Pc1.

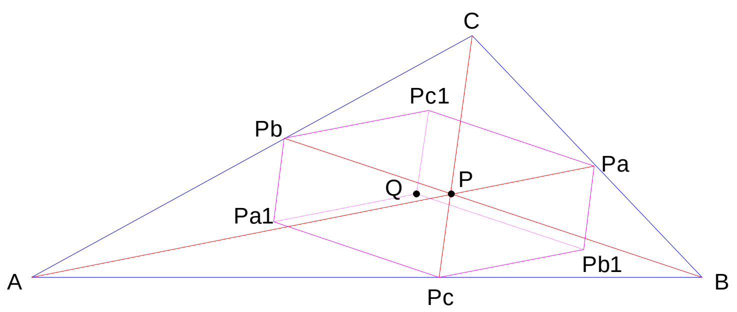

Fig. 4

Now lines through Pa1, Pb1, Pc1 and parallel to the cevians APa, BPb, CPc respectively, are concurrent at the point Q with barycentric coordinates f(x,y,z) : f(y,z,x) : f(z,x,y), where

$$f(x,y,z) = x*(y+z)*(y^2+z^2+x*(y+z)) \quad . \qquad (17)$$

Transformation defined by (17) maps the incenter I = a : b : c into the center X(2292) , and the incircle is mapped into a trefoil knot like curve shown in Fig. 5.

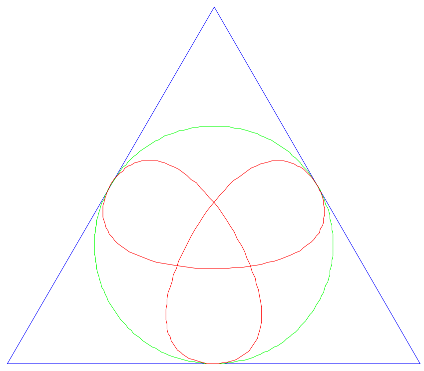

Fig. 5

In Fig.6 is shown another triangle center. Let P = x : y : z be an arbitrary point. The line through P and parallel to the side BC intersects sides AB, AC at points Cb, Bc respectively. Similarly we construct points Ac, Ca and Ba, Ab . The line through Ca and parallel to CP intersects the line through Ba and parallel to BP at a point A1. Similarly we construct points B1 and C1. Now lines AA1, BB1, CC1 are concurrent at a point Q with barycentric coordinates f(x,y,z) : f(y,z,x) : f(z,x,y), where

$$f(x,y,z) = x*(y+z)/(y+z-x) \quad . \qquad (18)$$

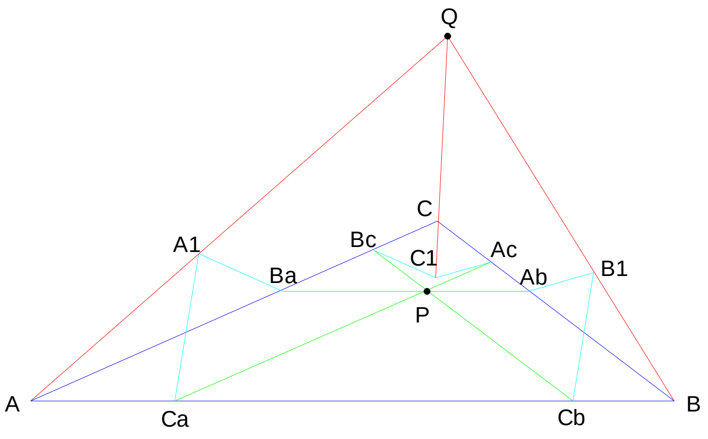

Fig. 6

The transformation defined by (18) maps the incenter X(1) into the center X(65).